

SUPERSYMMETRIC M2-BRANES WITH ENGLERT FLUXES,
AND
THE SIMPLE GROUP $\mathrm{PSL}(2, 7)$

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Abstract

A new class is introduced of M2-branes solutions of d=11 supergravity that include internal fluxes obeying Englert equation in 7-dimensions. A simple criterion for the existence of Killing spinors in such backgrounds is established. Englert equation is viewed as the generalization to d=7 of Beltrami equation defined in d=3 and it is treated accordingly. All 2-brane solutions of minimal d=7 supergravity can be uplifted to d=11 and have $\mathcal{N} \geq 4$ supersymmetry. It is shown that the simple group $\mathrm{PSL}(2, 7)$ is crystallographic in d=7 having an integral action on the A7 root lattice. By means of this point-group and of the T^7 torus obtained quotienting \mathbb{R}^7 with the A7 root lattice we were able to construct new M2 branes with Englert fluxes and $\mathcal{N} \leq 4$. In particular we exhibit here an $\mathcal{N} = 1$ solution depending on 4-parameters and admitting a large non abelian discrete symmetry, namely $G_{21} \equiv \mathbb{Z}_3 \ltimes \mathbb{Z}_7 \subset \mathrm{PSL}(2, 7)$. The dual $d = 3$ field theories have the same symmetries and have complicated non linear interactions.

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This paper is dedicated by the author to his very good and distinguished friend the Nobel Laureate Francois Englert, remembering the good time when he first announced in the Trieste Spring School his newly found solution of M-theory. The author hopes that Francois will like this new, unexpected Englert solution with $\mathcal{N} = 1$ supersymmetry.

1 Introduction

The finite group:

$$L_{168} \equiv \text{PSL}(2, \mathbb{Z}_7) \quad (1.1)$$

is the second smallest simple group after the alternating group A_5 which has 60 elements and coincides with the symmetry group of the regular icosahedron or dodecahedron. As anticipated by its given name, L_{168} has 168 elements: they can be identified with all the possible 2×2 matrices with determinant one whose entries belong to the finite field \mathbb{Z}_7 , counting them up to an overall sign. In projective geometry, L_{168} is classified as a *Hurwitz group* since it is the automorphism group of a Hurwitz Riemann surface, namely a surface of genus g with the maximal number $84(g-1)$ of conformal automorphisms². The Hurwitz surface pertaining to the Hurwitz group L_{168} is the Klein quartic [1], namely the locus \mathcal{K}_4 in $\mathbb{P}_2(\mathbb{C})$ cut out by the following quartic polynomial constraint on the homogeneous coordinates $\{x, y, z\}$:

$$x^3 y + y^3 z + z^3 x = 0 \quad (1.2)$$

Indeed \mathcal{K}_4 is a genus $g = 3$ compact Riemann surface and it can be realized as the quotient of the hyperbolic Poincaré plane \mathbb{H}_2 by a certain group Γ that acts freely on \mathbb{H}_2 by isometries.

The L_{168} group, which is also isomorphic to $\text{GL}(3, \mathbb{Z}_2)$, has received a lot of attention in Mathematics and it has important applications in algebra, geometry, and number theory: for instance, besides being associated with the Klein quartic, L_{168} is the automorphism group of the Fano plane. The reason why we consider L_{168} in this paper is associated with another property of this finite simple group which was proved fifteen years ago in [4], namely:

$$L_{168} \subset G_{2(-14)} \quad (1.3)$$

This means that L_{168} is a finite subgroup of the compact form of the exceptional Lie group G_2 and the 7-dimensional fundamental representation of the latter is irreducible upon restriction to L_{168} .

This fact is quite inspiring in the context of M-theory since it suggests possible connections with manifolds of G_2 -holonomy and alludes to scenarios where L_{168} plays some key role in compactifications $11 \rightarrow 4$ or in $M2$ -brane solutions.

These suggestions become much more circumstantial if we focus on the following linear equation:

$$\star d\mathbf{Y}^{[3]} = -\frac{\mu}{4} \mathbf{Y}^{[3]} \quad (1.4)$$

that implies

$$d \star \mathbf{Y}^{[3]} = 0 \quad (1.5)$$

and which we name Englert equation for the reasons we presently explain. In 1982, after the Freund

²Hurwitz's automorphisms theorem proved in 1893 [2] states that the order $|\mathcal{G}|$ of the group \mathcal{G} of orientation-preserving conformal automorphisms, of a compact Riemann surface of genus $g > 1$ admits the following upper bound $|\mathcal{G}| \leq 84(g-1)$

Rubin compactification of d=11 supergravity on the round 7-sphere had been found [5] and other similar solutions on different 7-manifolds had also been proposed [7],[6] Francois Englert introduced a new solution on the round 7-sphere [8] where the 4-form $\mathcal{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$ has two distinct non-vanishing parts, the first $e\epsilon_{\mu\nu\rho\sigma}$ living on the reduced space-time \mathcal{M}_4 and proportional to the epsilon symbol through a constant parameter (the Freund Rubin parameter e), the second \mathcal{F}_{IJKL} living on the internal \mathcal{M}_7 manifold and giving rise to a 3-form Φ and a dual 4-form $^*\Phi$ according to:

$$\begin{aligned} ^*\Phi &= \mathcal{F}_{IJKL} e^I \wedge e^J \wedge e^K \wedge e^L \\ \Phi &\equiv \frac{1}{24} \epsilon_{ABCIJKL} \mathcal{F}_{IJKL} e^A \wedge e^B \wedge e^C \end{aligned} \quad (1.6)$$

where e^I denote the vielbein 1-forms on \mathcal{M}_7 . The conditions that Φ has to satisfy in order to fit into an exact solution of $d = 11$ supergravity are the following ones:

$$^*d\Phi = 12e\Phi \quad ; \quad d^*\Phi = 0 \quad (1.7)$$

As one sees these conditions just coincide with what we named Englert equation: it suffices to set $\mathbf{Y}^{[3]} = \Phi$ and identify $\frac{\mu}{4} = -12e$. In [9], together with M. Trigiante I showed that the existence of a form Φ satisfying eq.s (1.7) is equivalent to the definition, introduced in [10], of manifolds \mathcal{M}_7 of weak G_2 -holonomy. This is the new phrasing, in modern parlance, of the notion originally introduced at the beginning of the eighties by the authors of [11] under the name of Englert manifolds. On Englert manifolds \mathcal{M}_7 we have non trivial solutions η of the following weak Killing spinor equation:

$$\mathcal{D}\eta = -\frac{3}{2} e \tau_I \eta e^I \quad (1.8)$$

where τ_I are a set of real gamma matrices:

$$\{\tau_I, \tau_J\} = -2\delta_{IJ} \quad (1.9)$$

It suffice to set:

$$\Phi \equiv \frac{1}{24} \epsilon_{ABCIJKL} (\eta^T \tau_{IJKL} \eta) e^A \wedge e^B \wedge e^C \quad (1.10)$$

and in force of eq.(1.8), Englert equation (1.6) is satisfied.

In the recent paper [12] it was observed by some of us that Englert equation (1.4) is the natural generalization to 7-dimensions of Beltrami equation:

$$\star d\mathbf{Y}^{[1]} = \mu \mathbf{Y}^{[1]} \quad (1.11)$$

which, introduced in 1889 [13] by the famous italian mathematician Eugenio Beltrami, has a distinguished and rich history in Mathematics, in particular in connection with hydrodynamics[14, 15] and with an important theorem on chaotic streamlines that was proved in the early seventies by Vladimir Arnold [16, 17]. For almost three decades a particular very simple solution of eq.(1.11), derived on a cubic 3-torus T^3 and named the ABC-flow, has been extensively investigated in the physical-mathematical literature [18] providing a working ground for both numerical and analytical studies. At the beginning of this year together with A.Sorin I presented a systematic algorithm [19]

for the solution of Beltrami equation on metric torii of the form:

$$T^3 \simeq \frac{\mathbb{R}^3}{\Lambda_{crys}} \quad (1.12)$$

where Λ_{crys} denotes a crystallographic lattice. The catch of such algorithm is the use of the Point Group $\mathbb{G}_{\text{Point}}$ of the lattice and of its orbits \mathfrak{S}_k in the dual momentum lattice $^*\Lambda_{crys}$. Summing on periodic functions associated with each of the momenta in a given orbit one obtains a solution of Beltrami equation that depends on a number of parameters which is uniquely determined by the length of the orbit. This solution can be later decomposed into irreducible representation either of the Point Group or of its Space-Group extensions that include also appropriate discrete translations. A complete classification of solutions of (1.11) was obtained in [19] for the case where the crystallographic lattice is the cubic one Λ_{cubic} and, consequently, the Point Group is the order 24 octahedral group O_{24} .

In a series of further papers [20, 21, 12] appeared this year, it was shown that Arnold-Beltrami Flows, namely one-forms $\mathbf{Y}^{[1]}$ satisfying Beltrami equation (1.11) can be used to construct fluxes in the transverse space to 2-brane solutions of $d = 7$ minimal supergravity [22, 24, 23] whose residual supersymmetries can also be counted [12].

As it was briefly sketched in [12], all these 2-branes solutions of $d = 7$ supergravity can be uplifted to $M2$ -branes solutions of $d=11$ supergravity and constitute particular cases of a more general class of $M2$ -branes that is the purpose of the present paper to describe.

The key issue is Englert equation (1.4) and the question whether we are able to solve it on 7-dimensional metric torii of the form:

$$T^7 \simeq \frac{\mathbb{R}^7}{\Lambda} \quad (1.13)$$

where Λ is some appropriate lattice in $d = 7$. Unfortunately little is known in Mathematics on the classification of finite subgroups of higher dimensional rotation groups and on the classification of crystallographic lattices in $d > 5$, yet some intuition and educated guesses can take us a long way ahead. Knowing for sure that eq.(1.3) holds true, it is tempting to guess that L_{168} is crystallographic in $d = 7$. This means that there should be a basis of vectors:

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_7 \quad (1.14)$$

in which the 7×7 matrices $D[\gamma]$ representing the elements of L_{168} have *integer valued* entries, have determinant one and are orthogonal with respect to the flat metric:

$$\eta_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j \quad (1.15)$$

namely:

$$\forall \gamma \in L_{168} \quad : \quad D[\gamma]^T \eta D[\gamma] = \eta \quad (1.16)$$

As we show in this paper such a guess is true. The appropriate crystallographic lattice Λ is the *root lattice* Λ_{root} of the A_7 simple Lie algebra and the invariant metric η is the corresponding Cartan matrix \mathcal{C} . Thanks to this we were able to extend the algorithms of [19] to Englert equation (1.4) and construct rich families of its solutions associated with L_{168} orbits in the A_7 weight lattice. On the other hand each solution of (1.4) gives rise to an exact $M2$ -brane solution of M-theory. The new vast bestiary of branes has now to be investigated for properties and applications.

One important point to stress is that the choice of the lattice $\Lambda \subset \mathbb{R}^7$ is equivalent to the choice of a flat metric η_{ij} as it is mentioned in eq. (1.15). The space of flat metrics on T^7 is a well known coset:

$$\mathcal{M}_{moduli} [T^7] = \frac{SL(7, \mathbb{R})}{O(7)} \quad (1.17)$$

hence when we choose a crystallographic group \mathbb{G}_{point} and a lattice Λ we just choose a particular point in the moduli space (1.17) of T^7 and the metric η_{ij} should be regarded as an item in the M2-brane solution that we construct.

Obviously, just as in $d = 3$, also in $d = 7$ there is not a single crystallographic lattice and the A7-root lattice with the point group L_{168} is not the only choice. One can find solutions of Englert equation also on torii identified by different lattices. For instance the uplifting to $d = 11$ of the solutions with Arnold-Beltrami fluxes that we found in $d = 7$ corresponds to the choice of a 7-dimensional lattice of the following form:

$$\Lambda^{[7]} = \Lambda_{cubic}^{[3]} \otimes \Lambda_{cubic}^{[4]} \quad (1.18)$$

which can be easily proved to admit no crystallographic embedding in the A7-root lattice. Hence the 2-branes solutions of $d = 7$ minimal supergravity with Arnold-Beltrami fluxes are uplifted to $d = 11$ in different points of the T^7 moduli space (1.17).

The choice of the A7-lattice and of the crystallographic group L_{168} has one distinguished advantage. The presence of a \mathbb{Z}_7 subgroup and the immersion in $G_{2(-14)}$ appears to be the basis for the existence of $\mathcal{N} = 1$ supersymmetric M2-brane solutions with Englert fluxes.

In this paper we discuss the general criterion for the existence of Killing spinors in M2-brane solutions with Englert fluxes. We show that the uplifting of Arnold-Beltrami solutions of $d = 7$ supergravity to $d = 11$ has always a large residual supersymmetry, namely $\mathcal{N} \geq 4$ in $d = 3$ for a total of $\#$ of *supercharges* ≥ 8 .

On the other hand using the A7-lattice, the number of Killing spinors is bounded from below by zero: $\mathcal{N} \geq 0$.

In this paper we explicitly construct a solution with $\mathcal{N} = 1$ which has a discrete symmetry group of order 21.

We do not feel it necessary to insert the traditional illustration of the content of the various sections since it is evident from the table of contents at the beginning of this paper.

2 M2-branes with Englert fluxes

In this section we consider the general form of M2-brane solutions with fluxes and their relation with Englert equation. To this effect let us consider the effective low energy lagrangian of M -theory, namely $d = 11$ supergravity for which we utilize the geometric rheonomic formulation of [25, 26].

2.1 Summary of d=11 supergravity in the rheonomy framework

The complete set of curvatures defining the relevant Free Differential Algebra is given below ([25, 26]):

$$\begin{aligned}
\mathfrak{T}^a &= \mathcal{D}V^a - i\frac{1}{2}\bar{\psi} \wedge \Gamma^a \psi \\
\mathfrak{R}^{ab} &= d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} \\
\rho &= \mathcal{D}\psi \equiv d\psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab} \psi \\
\mathbf{F}^{[4]} &= d\mathbf{A}^{[3]} - \frac{1}{2}\bar{\psi} \wedge \Gamma_{ab} \psi \wedge V^a \wedge V^b \\
\mathbf{F}^{[7]} &= d\mathbf{A}^{[6]} - 15\mathbf{F}^{[4]} \wedge \mathbf{A}^{[3]} - \frac{15}{2}V^a \wedge V^b \wedge \bar{\psi} \wedge \Gamma_{ab} \psi \wedge \mathbf{A}^{[3]} \\
&\quad - i\frac{1}{2}\bar{\psi} \wedge \Gamma_{a_1\dots a_5} \psi \wedge V^{a_1} \wedge \dots \wedge V^{a_5}
\end{aligned} \tag{2.1}$$

From their very definition, by taking a further exterior derivative one obtains the Bianchi identities:

$$\mathcal{D}\mathfrak{R}^{ab} = 0 \tag{2.2}$$

$$\mathcal{D}\mathfrak{T}^a + \mathfrak{R}^{ab} \wedge V_b - i\bar{\psi} \wedge \Gamma^a \rho = 0 \tag{2.3}$$

$$\mathcal{D}\rho + \frac{1}{4}\Gamma^{ab}\psi \wedge \mathfrak{R}^{ab} = 0 \tag{2.4}$$

$$d\mathbf{F}^{[4]} - \bar{\psi} \wedge \Gamma_{ab} \rho \wedge V^a \wedge V^b + \bar{\psi} \Gamma_{ab} \psi \wedge \mathfrak{T}^a \wedge V^b = 0 \tag{2.5}$$

$$\begin{aligned}
&d\mathbf{F}^{[7]} - i\bar{\psi} \wedge \Gamma_{a_1\dots a_5} \rho \wedge V^{a_1} \wedge \dots \wedge V^{a_5} \\
&\quad - \frac{5}{3}i\bar{\psi} \wedge \Gamma_{a_1\dots a_5} \psi \wedge \mathfrak{T}^{a_1} \wedge V^{a_2} \wedge \dots \wedge V^{a_5} \\
&\quad - 15\bar{\psi} \wedge \Gamma_{ab} \rho \wedge V^a \wedge V^b \wedge \mathbf{F}^{[4]} - 15\mathbf{F}^{[4]} \wedge \mathbf{F}^{[4]} = 0
\end{aligned} \tag{2.6}$$

There is a unique rheonomic parametrization of the curvatures (2.1) which solves the Bianchi identities and it is the following one:

$$\begin{aligned}
\mathfrak{T}^a &= 0 \\
\mathbf{F}^{[4]} &= F_{a_1\dots a_4} V^{a_1} \wedge \dots \wedge V^{a_4} \\
\mathbf{F}^{[7]} &= \frac{1}{84}F^{a_1\dots a_4} V^{b_1} \wedge \dots \wedge V^{b_7} \epsilon_{a_1\dots a_4 b_1\dots b_7} \\
\rho &= \rho_{a_1 a_2} V^{a_1} \wedge V^{a_2} - i\frac{1}{3}(\Gamma^{a_1 a_2 a_3} \psi \wedge V^{a_4} + \frac{1}{8}\Gamma^{a_1\dots a_4 m} \psi \wedge V^m) F^{a_1\dots a_4} \\
\mathfrak{R}^{ab} &= R^{ab}_{cd} V^c \wedge V^d + i\rho_{mn} \left(\frac{1}{2}\Gamma^{abmn} - \frac{2}{9}\Gamma^{mn[a} \delta^{b]c} + 2\Gamma^{ab[m} \delta^{n]c} \right) \psi \wedge V^c \\
&\quad + \bar{\psi} \wedge \Gamma^{mn} \psi F^{mnab} + \frac{1}{24}\bar{\psi} \wedge \Gamma^{abc_1\dots c_4} \psi F^{c_1\dots c_4}
\end{aligned} \tag{2.7}$$

The expressions (2.7) satisfy the Bianchi.s provided the space-time components of the curvatures satisfy the following constraints

$$0 = \mathcal{D}_m F^{mc_1 c_2 c_3} + \frac{1}{96} \epsilon^{c_1 c_2 c_3 a_1 a_8} F_{a_1\dots a_4} F_{a_5\dots a_8} \tag{2.8}$$

$$0 = \Gamma^{abc} \rho_{bc} \tag{2.9}$$

$$R^{am}_{cm} = 6 F^{ac_1 c_2 c_3} F^{bc_1 c_2 c_3} - \frac{1}{2} \delta_b^a F^{c_1\dots c_4} F^{c_1\dots c_4} \tag{2.10}$$

which are the space-time field equations.

2.2 M2-brane solutions with $\mathbb{R}_+ \times T^7$ in the transverse dimensions

Among the many possible solutions of the field equations (2.8-2.10) we are interested in those that describe $M2$ -branes of the following sort.

Inspired by our previous results in $d = 7$ [21, 12], we give the 11-dimensional manifold the following topology:

$$\mathcal{M}_{11} = \text{Mink}_{1,2} \times \mathbb{R}_+ \times T^7 \quad (2.11)$$

where $\text{Mink}_{1,2}$ is Minkowski space in $1 + 2$ dimensions and represents the world-volume of the $M2$ -brane, while T^7 is a flat compact seven-torus. $\mathbb{R}_+ \times T^7$ is the eight-dimensional space transverse to the brane.

Then, according to the general rules of brane-chemistry (see for instance [27], page 288 and following ones), we introduce the following $d = 11$ metric:

$$ds_{11}^2 = H(y)^{-\frac{4\tilde{d}}{9\Delta}} (d\xi^\mu \otimes d\xi^\nu \eta_{\mu\nu}) - H(y)^{\frac{4d}{9\Delta}} (dy^I \otimes dy^J \delta_{IJ}) \quad (2.12)$$

where:

$$\xi^\mu \quad ; \quad \mu = \underline{0}, \underline{1}, \underline{2} \quad (2.13)$$

are the coordinates on $\text{Mink}_{1,2}$, while:

$$y^I \quad ; \quad I = 0, 1, 2, \dots, 7 \quad (2.14)$$

are the coordinates of the 8-dimensional transverse space. Since in $d = 11$ there is no dilaton we have

$$\Delta = 2 \frac{\tilde{d}d}{9} = 2 \frac{6 \times 3}{9} = 4 \quad ; \quad d = 3; \quad \tilde{d} = 6 \quad (2.15)$$

and the appropriate $M2$ ansatz for the metric becomes:

$$ds_{11}^2 = H(y)^{-\frac{2}{3}} (d\xi^\mu \otimes d\xi^\nu \eta_{\mu\nu}) - H(y)^{\frac{1}{3}} (dy^I \otimes dy^J \delta_{IJ}) \quad (2.16)$$

Because of the chosen topology of the transverse space it is convenient to set:

$$y^0 = U \in \mathbb{R}_+ \quad ; \quad y^i = x^i \in T^7 \quad (i = 1, \dots, 7) \quad (2.17)$$

The next point is to choose an appropriate ansatz for the three-form $\mathbf{A}^{[3]}$. We set:

$$\mathbf{A}^{[3]} = \frac{2}{H(y)} \Omega^{[3]} + e^{-\mu U} \mathbf{Y}^{[3]} \quad (2.18)$$

where:

$$\Omega^{[3]} = \frac{1}{6} \epsilon_{\mu\nu\rho} d\xi^\mu \wedge d\xi^\nu \wedge d\xi^\rho \quad (2.19)$$

$$\mathbf{Y}^{[3]} = Y_{ijk}(x) dx^i \wedge dx^j \wedge dx^k \quad (2.20)$$

The essential point in the above formula is that the antisymmetric tri-tensor $Y_{ijk}(x)$ depends only on the coordinates x of the seven-torus T^7 .

2.2.1 Analysis of Maxwell equation

With the above data we are in a position to work out the explicit form of the 4-form field strength, its intrinsic anholonomic components F_{abcd} and insert them into the field equation (2.8). An essential ingredient in this calculation is provided by the transcription of the metric (2.16) and by the calculation of the spin connection.

We set:

$$V^a = H(y)^{-\frac{1}{3}} d\xi^a \quad (2.21)$$

$$V^I = H(y)^{\frac{1}{6}} dy^I \quad (2.22)$$

and we obtain:

$$\omega^{ab} = 0 \quad (2.23)$$

$$\omega^{aI} = -\frac{1}{3} H^{-\frac{7}{6}} V^a \partial^I H(y) \quad (2.24)$$

$$\omega^{IJ} = \frac{1}{6} H^{-\frac{7}{6}} (\partial^I H(y) V^J - \partial^J H(y) V^I) \quad (2.25)$$

$$(2.26)$$

With the ansatz (2.18) the non vanishing components of the 4-form $\mathbf{F}^{[4]}$ are the following ones:

$$F_{abcI} = \frac{1}{12} H(y)^{-\frac{7}{6}} \partial_I H(y) \quad (2.27)$$

$$F_{0ijk} = -\frac{\mu}{4} e^{-\mu U} H(y)^{-\frac{2}{3}} Y_{ijk} \quad (2.28)$$

$$F_{ijkl} = H(y)^{-\frac{2}{3}} e^{-\mu U} \partial_i Y_{jkl} \quad (2.29)$$

Then we can easily verify that the Maxwell field equation (2.8) is satisfied provided the following two differential constraints hold true:

$$\square_{\mathbb{R}_+ \times T^7} H(y) = \frac{\mu}{4} e^{-2\mu U} \epsilon^{ijklmnr} \partial_i Y_{jkl} Y_{mnr} \quad (2.30)$$

$$\frac{1}{4!} \epsilon^{pqrijkl} \partial_i Y_{jkl} = -\frac{\mu}{4} Y_{pqr} \quad (2.31)$$

The two equations admit the following index-free rewriting:

$$\square_{\mathbb{R}_+ \times T^7} H(y) = -\frac{3\mu^2}{2} e^{-2\mu U} \|\mathbf{Y}\|^2 \equiv J(y) \quad (2.32)$$

$$\star_{T^7} d\mathbf{Y}^{[3]} = -\frac{\mu}{4} \mathbf{Y}^{[3]} \quad (2.33)$$

As we see eq.(2.33) is the generalization to a 7-dimensional torus of Beltrami equation on the three-dimensional one: it is just Englert equation (1.4) discussed in the introduction.

3 The Killing spinor equation of M2-branes with Englert fluxes

In order to analyze the structure of the Killing spinor equation in the background of the M2-branes with Englert fluxes, introduced in the previous section, we need a basis of gamma matrices that is well-adapted to the splitting of the 11-dimensional manifold, namely:

$$\mathcal{M}_{11} = \underbrace{\text{Mink}_{1,2}}_{d=3 \text{ brane world volume}} \times \underbrace{\left(\mathbb{R}_+ \otimes \underbrace{\text{T}^7}_{d=7} \right)}_{d=8 \text{ transverse space}} \quad (3.1)$$

Such a well adapted basis is provided by the following nested hierarchy.

3.1 Gamma matrices

At the bottom of the hierarchy we have the Pauli matrices.

Pauli matrices. We use the following conventions:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \quad (3.2)$$

Gamma matrices on the d=3 world-volume. Next we construct the set of 2×2 gamma matrices in d=3 in the following way

$$\{\gamma_{\underline{a}}, \gamma_{\underline{b}}\} = 2\eta_{\underline{a}\underline{b}} \quad ; \quad \gamma = \{\sigma_2, i\sigma_1, i\sigma_3\} \quad (3.3)$$

Gamma matrices in d=7 In d=7 we choose gamma matrices that are real and antisymmetric and fulfill the following Clifford algebra:

$$\{\tau_i, \tau_j\} = -2\delta_{ij} \quad (3.4)$$

The explicit basis utilized is that one where we express the τ -matrices in terms of ϕ_{ijk} , namely of the G_2 -invariant three-tensor:

$$\begin{aligned} (\tau_i)_{jk} &= \phi_{ijk} \\ (\tau_i)_{j8} &= \delta_{ij} \quad ; \quad (\tau_i)_{8j} = -\delta_{ij} \end{aligned} \quad (3.5)$$

The explicit form of the ϕ_{ijk} tensor will be given in eq.(5.13) and it is the one well-adapted to the immersion of the discrete group which acts crystallographically on T^7 into the compact G_2 Lie group, namely according to the canonical immersion $L_{168} \longrightarrow G_{2(-14)}$.

Gamma matrices in d=8 Because of our splitting $11 = 3 \oplus 1 \oplus 7$ we need also the gamma matrices in d=8 corresponding to the transverse space to the M2-brane, namely $\mathbb{R}_+ \otimes T^7$. We choose the following Clifford algebra:

$$\{T_I, T_J\} = -2\delta_{IJ} \quad (3.6)$$

and we utilize the following explicit realization:

$$\begin{aligned} T_0 &= i\sigma_2 \otimes \mathbf{1}_{8 \times 8} \\ T_i &= \sigma_1 \otimes \tau_i \\ T_9 &= \sigma_3 \otimes \mathbf{1}_{8 \times 8} \end{aligned} \quad (3.7)$$

The last matrix is the d=8 chirality operator which plays an important role in the discussion of the Killing spinor equation.

Gamma matrices in d=11 At the top of the hierarchy we have the d=11 gamma matrices, obeying the following Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \quad (3.8)$$

For them we utilize the following explicit realization:

$$\begin{aligned} \Gamma_a &= \gamma_a \otimes T_9 \\ \Gamma_I &= \mathbf{1}_{2 \times 2} \otimes T_I \end{aligned} \quad (3.9)$$

With these choices the charge conjugation matrix, takes the following form:

$$\begin{aligned} \mathfrak{C} &= i\sigma_2 \otimes \mathbf{1}_{2 \times 2} \otimes \mathbf{1}_{8 \times 8} \\ \mathfrak{C} \Gamma_a \mathfrak{C}^{-1} &= -\Gamma_a^T \end{aligned} \quad (3.10)$$

Equipped with this set of properly chosen gamma matrices we can turn to the investigation of the Killing spinor equation.

3.2 The tensor structure of the Killing spinor equation

The rheonomic solution of the d=11 Bianchi identities (see eq.(2.7)) allows us to write the Killing spinor equation in the following general form:

$$\mathcal{D}\xi - \frac{i}{3}\Gamma^{abc}V^d F_{abcd}\xi - \frac{i}{24}\Gamma_{abcd}F^{abcd}V^f \xi = 0 \quad (3.11)$$

where

$$\mathcal{D}\xi \equiv d\xi - \frac{1}{4}\omega^{ab}\Gamma_{ab}\xi \quad (3.12)$$

is the Lorentz covariant differential in $d = 11$.

Equation (3.11) can be usefully rewritten as follows:

$$\nabla\xi \equiv d\xi + \Omega\xi = 0 \quad (3.13)$$

where Ω is a generalized connection in the 32-dimensional spinor space, defined as follows:

$$\Omega \equiv \Theta_L + \Theta_1^{[F]} + \Theta_2^{[F]} \quad (3.14)$$

In the above equation we have introduced the following definitions:

$$\begin{aligned}
\Theta_L &\equiv -\frac{1}{4}\omega^{ab}\Gamma_{ab} \\
\Theta_1^{[F]} &\equiv -\frac{i}{3}\Gamma^{abc}V^dF_{abcd} \\
\Theta_2^{[F]} &\equiv -\frac{i}{24}\Gamma_{abcd}F^{abcd}V^f
\end{aligned} \tag{3.15}$$

Next let us make another splitting of the overall generalized connection:

$$\Omega = \Omega_H + \Omega_Y \tag{3.16}$$

where Ω_H depends only on the (inhomogeneous)-harmonic function H and it is obtained from Ω by setting $Y_{ijk} \rightarrow 0$. Instead, the other part Ω_Y , is just the difference and it depends linearly on Y_{ijk}

3.2.1 M2-branes without Englert fluxes: tensor structure of Ω_H

Introducing the following operators:

$$V \circ \gamma = V^a \gamma_a \tag{3.17}$$

$$\mathbb{P}_\pm = \frac{1}{2}(\mathbf{1}_{16} \pm T_9) \tag{3.18}$$

$$\partial H \circ T = \frac{1}{3}H^{-\frac{7}{6}}\partial_I H T^I \tag{3.19}$$

$$V \diamond \partial H \circ T = -\frac{1}{12}H^{-\frac{7}{6}}V_{[I}\partial_{J]} H T^{IJ} \tag{3.20}$$

$$\mathbf{d}H = \frac{1}{6}H^{-\frac{7}{6}}\sum_{I=1}^8\partial_I H V^I \tag{3.21}$$

we get that the H -part of the generalized connection has the following tensor structure:

$$\Omega_H = V \circ \gamma \otimes \partial H \circ T \mathbb{P}_- + \mathbf{1}_2 \otimes V \diamond \partial H \circ T \mathbb{P}_- + \mathbf{1}_2 \otimes \mathbf{d}H T_9 \tag{3.22}$$

From equation (3.22) one readily derives the form of the Killing spinors for pure M2-brane solutions. Writing the 32 component Killing spinor as a tensor product:

$$\xi = \epsilon \otimes \chi \tag{3.23}$$

we find that, in the absence of Y -fields, the Killing spinor equation is satisfied provided:

$$T_9 \chi = \chi \Rightarrow \mathbb{P}_- \chi = 0 \tag{3.24}$$

$$\chi = H^{-\frac{1}{6}} \chi_0 \tag{3.25}$$

where H is the (inhomogeneous)-harmonic function appearing in the metric (2.12) and χ_0 is a constant spinor with commuting components. Indeed, in view of our 2-brane interpretation of these backgrounds, we assume that the two-component spinors ϵ are the anticommuting objects.

Using the tensor structure of the d=8 T-matrices we set:

$$\chi = \kappa \otimes \lambda \quad (3.26)$$

where κ is a two component spinor:

$$\kappa = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} \quad (3.27)$$

with commuting components, while λ is an eight-component spinor:

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix} \quad (3.28)$$

also with commuting components.

In this language the most general 32-component spinor has the form:

$$\xi = \epsilon \otimes \kappa \otimes \lambda \quad (3.29)$$

and the general solution for the Killing spinor at $Y_{ijk} = 0$ is obtained by setting:

$$\kappa_2 = 0 \quad ; \quad \kappa_1 = H^{-\frac{1}{6}} \quad (3.30)$$

This shows that the M2-branes without Englert-fluxes preserve 16 supersymmetries, namely $\frac{1}{2}$ of the total SUSY.

3.2.2 M2-branes with Englert fluxes: tensor structure of Ω_Y

We come next to analyze the structure of the Y-part of the connection Ω_Y .

We begin by introducing two $d = 7$ operators constructed with the Englert field Y_{ijk} , the flat 8-dimensional vielbein $\hat{V}^I \equiv dy^I$ and the τ -matrices:

$$\mathcal{B} \equiv \tau_{ijk} Y_{ijk} \quad ; \quad \mathcal{T} = \hat{V}^i \tau_i \quad (3.31)$$

In terms of these operators we get:

$$\begin{aligned}\Omega_Y &= \frac{\mu}{4} e^{-U\mu} H^{-2/3} \left(\frac{1}{3} i V \circ \gamma \otimes \mathbb{P}_+ \otimes \mathcal{B} \right. \\ &\quad + \frac{1}{6} i \mathbf{1} \otimes \left(\sigma_1 + \frac{i}{2} \sigma_2 \right) \otimes [\mathcal{B}, \mathcal{T}] + \frac{1}{6} i \mathbf{1} \otimes \left(\frac{1}{2} \sigma_1 + i \sigma_2 \right) \otimes \{\mathcal{B}, \mathcal{T}\} \\ &\quad \left. + \frac{1}{3} i \mathbf{1} \otimes \mathbb{P}_+ \otimes \mathcal{B} \hat{V}^0 \right) \end{aligned} \quad (3.32)$$

where $[\mathcal{B}, \mathcal{T}]$ and $\{\mathcal{B}, \mathcal{T}\}$ respectively denote the commutator and the anti-commutator of the two mentioned operators. Observing the structure of the connection Ω_Y displayed in eq.(3.32) we can rewrite it in the following more explicit form:

$$\begin{aligned}\Omega_Y &= i \frac{1}{12} \mu e^{-U\mu} H^{-2/3} \times \\ &\quad \left[V \circ \gamma \begin{pmatrix} 2\mathcal{B} & 0 \\ 0 & 0 \end{pmatrix} + \mathbf{1} \otimes \begin{pmatrix} \hat{V}^0 \mathcal{B} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2} \mathbf{1} \otimes \begin{pmatrix} 0 & 3\mathcal{B}\mathcal{T} \\ -\mathcal{T}\mathcal{B} & 0 \end{pmatrix} \right] \end{aligned} \quad (3.33)$$

Eq.(3.33) reveals the mechanism behind the preservation of supersymmetry by M2-branes with Englert fluxes. Writing the candidate Killing spinor in the tensor product form (3.29) we see that the connection Ω_Y annihilates it if $\kappa = \begin{pmatrix} H^{-\frac{1}{6}} \\ 0 \end{pmatrix}$, as we already established from consideration of the H-part of the connection, and if the 8-component λ is a null-vector of \mathcal{B} :

$$\mathcal{B} \lambda = 0 \quad (3.34)$$

This is the only possibility to integrate the Killing spinor equation. Indeed the term with $V \circ \gamma$ which mixes the internal coordinates with the world volume ones has to vanish since it cannot be compensated in any other way. This implies eq.(3.34). The magic thing is that because of the precise values of the coefficients provided by the rheonomic solution of Bianchi identities in d=11, the combinations of commutators and anticommutators appearing in eq.(3.32) just produce the structure in eq.(3.33). In this way the condition (3.34) suffices to annihilate also the action of the other terms in the connection.

In conclusion M2-branes with Englert fluxes preserve part of the Killing spinors existing in the case of $Y = 0$ if and only if the operator \mathcal{B} has a non trivial Null-Space, namely if the Rank of \mathcal{B} is < 8 . Every λ satisfying (3.34) corresponds to a preserved supersymmetry.

4 Uplifting of 7-dimensional Arnold-Beltrami flux 2-branes to Englert M2-branes

In order to uplift the solutions of $d = 7$ minimal supergravity to the Englert M2-branes described in section. 2.2 we split the 7-torus as it follows:

$$\mathbb{T}^7 = \mathbb{T}^3 \otimes \mathbb{T}^4 \quad (4.1)$$

and we name $\mathbf{x}_\parallel = x^i$, ($i, j, k = 1, 2, 3$) the coordinates of the 3-tours and $\mathbf{x}_\perp = x^\alpha$, ($\alpha, \beta, \gamma = 4, 5, 6, 7$) the coordinates of the 4-torus. The next step consists of setting:

$$\begin{aligned}\mathbf{Y}^{[3]}[\mathbf{W}] &= \mathbf{W}^\Lambda \wedge \mathbb{K}^\Lambda \\ \mathbb{K}^\Lambda &= J_{\alpha\beta}^{\Lambda|-} dx^\alpha \wedge dx^\beta\end{aligned}\quad (4.2)$$

where \mathbf{W}^Λ denotes a triplet of Beltrami one-forms satisfying Beltrami equation on the 3-torus:

$$\star_{T^3} d\mathbf{W}^\Lambda = \mu \mathbf{W}^\Lambda \quad (4.3)$$

while $J_{\alpha\beta}^{\Lambda|-}$ is the triplet of anti-self-dual 't Hooft matrices, leading to:

$$\mathbb{K}^\Lambda = \begin{pmatrix} 2dx^6 \wedge dx^7 - 2dx^4 \wedge dx^5 \\ 2dx^4 \wedge dx^6 + 2dx^5 \wedge dx^7 \\ 2dx^5 \wedge dx^6 - 2dx^4 \wedge dx^7 \end{pmatrix} \quad (4.4)$$

Any triplet of Beltrami vector fields \mathbf{W}^Λ satisfying eq.(4.3) produces an Englert 3-form $\mathbf{Y}^{[3]}[\mathbf{W}]$ satisfying Englert equation (2.31) with the same μ . Equation (2.32) reduces to:

$$\square_{\mathbb{R}_+ \times T^3} H(U, \mathbf{x}_\parallel) = -6\mu^2 e^{-2\mu U} \sum_{\Lambda=1}^3 \|\mathbf{W}^\Lambda\|^2 \quad (4.5)$$

and the relations of the $d=7$ dilaton and metric with the $d=11$ metric are the following ones:

$$\begin{aligned}\phi &= -\frac{2}{5} \log [H(U, \mathbf{x}_\parallel)] \\ ds_7^2 &= H(U, \mathbf{x}_\parallel)^{-\frac{2}{5}} (d\xi^\mu \otimes d\xi^\nu \eta_{\mu\nu}) - H(U, \mathbf{x}_\parallel)^{\frac{3}{5}} (dU^2 + d\mathbf{x}_\parallel^2) \\ ds_{11}^2 &= \exp\left[\frac{2}{3}\phi\right] ds_7^2 - \exp\left[-\frac{5}{6}\phi\right] d\mathbf{x}_\perp^2 \\ &\Downarrow \\ ds_{11}^2 &= H(U, \mathbf{x}_\parallel)^{-\frac{2}{3}} (d\xi^\mu \otimes d\xi^\nu \eta_{\mu\nu}) - H(U, \mathbf{x}_\parallel)^{\frac{1}{3}} (dU^2 + d\mathbf{x}_\parallel^2 + d\mathbf{x}_\perp^2)\end{aligned} \quad (4.6)$$

4.1 Uplifting of Killing spinors

We wonder whether the supersymmetry possibly preserved by the Arnold-Beltrami flux 2-brane in $d=7$ is preserved by its uplifting to an M2-brane with Englert fluxes. The answer is clearly yes and actually the $d=11$ preserved SUSY is larger. In addition to the Killing q -charges preserved in $d=7$ one has 8 further ones corresponding to $\frac{1}{2}$ of the supersymmetries that are truncated away in the consistent truncation to minimal $d=7$ supergravity. We illustrate this mechanism with the explicit consideration of an example.

4.1.1 Uplift of the Arnold Beltrami flux 2-brane with $\mathcal{N} = 2$ SUSY

In [12] it was shown that the following triplet of Beltrami vector fields:

$$\mathbf{W}^\Lambda = \begin{pmatrix} dx^1 \cos 2\pi x^3 - dx^2 \sin 2\pi x^3 \\ dx^2 - \cos 2\pi x^3 - dx^1 \sin 2\pi x^3 \\ 0 \end{pmatrix} \quad (4.7)$$

leads to a d=7 supergravity solution preserving $\mathcal{N} = 2$ supersymmetry. Inserted into eq.(4.2), the vector field (4.7) produces an Englert field $Y_{ijk}[\mathbf{W}]$ which satisfies Englert eq.(1.4) with $\mu = 2\pi$ and admits, as solution of the inhomogeneous harmonic equation (2.31) the following function:

$$H(y) = 1 - e^{-4\pi U} \quad (4.8)$$

Inserted into eq.(3.31) the Englert field $Y_{ijk}[\mathbf{W}]$ produces the following \mathcal{B} -operator:

$$\mathcal{B}[\mathbf{W}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 \sin[2\pi x^3] & 0 & -4 \cos[2\pi x^3] & -4 \sin[2\pi x^3] & 0 & -4 \cos[2\pi x^3] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 \cos[2\pi x^3] & 0 & 4 \sin[2\pi x^3] & -4 \cos[2\pi x^3] & 0 & 4 \sin[2\pi x^3] \\ 0 & 0 & -4 \sin[2\pi x^3] & 0 & -4 \cos[2\pi x^3] & -4 \sin[2\pi x^3] & 0 & -4 \cos[2\pi x^3] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 \cos[2\pi x^3] & 0 & 4 \sin[2\pi x^3] & -4 \cos[2\pi x^3] & 0 & 4 \sin[2\pi x^3] \end{pmatrix} \quad (4.9)$$

The rank of the above matrix is 2 and the most general vector in the 6-dimensional null-space has the following form:

$$\zeta = \begin{pmatrix} \zeta_6 \\ \zeta_5 \\ -\zeta_3 \\ \zeta_4 \\ -\zeta_1 \\ \zeta_3 \\ \zeta_2 \\ \zeta_1 \end{pmatrix} ; \quad \mathcal{B}[\mathbf{W}] \zeta = 0 \quad (4.10)$$

This leads to the conclusion that:

$$\xi = \epsilon \otimes \begin{pmatrix} H^{-\frac{1}{6}} \\ 0 \end{pmatrix} \otimes \zeta \quad (4.11)$$

is a Killing spinor for the constructed M2-brane solution, namely we have a total of 12 preserved supercharges. From a $d=3$ point of view it means that the brane-volume theory has $\mathcal{N} = 6$ supersymmetry. How to understand this result we already outlined above. Two out of the six are the supersymmetries that are preserved in the theory obtained from $d=7$ supergravity. The other four are the supersymmetries corresponding to the gravitinos that are truncated away in $d=7$, when we reduce from maximal to minimal supergravity. From the point of view of the $d=3$ world volume theory, the multiplets arising from $d=7$ minimal supergravity are smaller since they are associated with the 4 transverse coordinates of $\mathbb{R}_+ \times T^3$ and with their fermionic superpartners. These multiplets support $\mathcal{N} = 2$ supersymmetry. The multiplets arising from the complete $d=11$ theory are bigger since they are associated with the 8-transverse coordinates of $\mathbb{R}_+ \times T^3 \otimes T^4$ and they support $\mathcal{N} = 6$ supersymmetry.

In conclusion we can establish the following general rule for the number of supersymmetries preserved by M2-branes with Englert fluxes that are uplift of the Arnold Beltrami 2-branes in $d = 7$:

$$\mathcal{N}_{d=11} = \mathcal{N}_{d=7} + 4 \quad (4.12)$$

As already emphasized in the introduction, the above discussion of supersymmetry is quite relevant for the $d = 3$ field theories on the brane world volume.

- A) Pure M2-branes without Englert fluxes produce a $d = 3$ theory with $\mathcal{N} = 8$ supersymmetry. This theory is essentially a free one containing 8 scalars and 8 fermions. In the case of $\text{AdS}_4 \times S^7$ compactifications it is the theory of the $\text{OSp}(8|4)$ -singleton [31]. Here the background geometry is different.
- B) Introducing Englert fluxes corresponds to the inclusion of non trivial interactions among the fields living on the brane that are just the transverse coordinates and their fermionic partners. The world volume fields now arrange into supermultiplets according with residual supersymmetry dictated by the number of Killing spinors.
- C) In the case of theories obtained from Arnold-Beltrami fluxes, the supersymmetry is at least $\mathcal{N} = 4$. However the $d = 3$ theory can be consistently truncated by removing 4 bosonic fields with their superpartners. In this way one is left with the $d = 3$ theory one might directly construct from the $d = 7$ 2-branes solution with Arnold Beltrami fluxes.
- D) We are interested in Englert fluxes which lead to $d = 3$ theories having supersymmetry $\mathcal{N} \leq 4$, in particular $\mathcal{N} = 1$ or $\mathcal{N} = 0$, where, possibly, no consistent truncation exists to smaller multiplets. For this reason we proceed to the consideration of the specially promising crystallographic group introduced in eq.(1.1).

5 The simple group $L_{168} = \text{PSL}(2, \mathbb{Z}_7)$

For the reasons outlined at the end of the previous section we turn next our attention to the simple group (1.1) and to its crystallographic action in $d = 7$. The Hurwitz simple group L_{168} is abstractly presented as follows:

$$L_{168} = \left(R, S, T \parallel R^2 = S^3 = T^7 = RST = (TSR)^4 = \mathbf{e} \right) \quad (5.1)$$

and, as its name implicitly advocates, it has order 168:

$$|L_{168}| = 168 \quad (5.2)$$

The elements of this simple group are organized in six conjugacy classes according to the scheme displayed below:

Conjugacy class	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3	\mathcal{C}_4	\mathcal{C}_5	\mathcal{C}_6
representative of the class	\mathbf{e}	R	S	TRS	T	SR
order of the elements in the class	1	2	3	4	7	7
number of elements in the class	1	21	56	42	24	24

$$(5.3)$$

As one sees from the above table (5.3) the group contains elements of order 2, 3, 4 and 7 and there are two inequivalent conjugacy classes of elements of the highest order. According to the general theory of finite groups, there are 6 different irreducible representations of dimensions 1, 6, 7, 8, 3, 3, respectively. The character table of the group L_{168} can be found in the mathematical literature, for instance in the book [28]. It reads as follows:

Representation	\mathcal{C}_1	\mathcal{C}_2	\mathcal{C}_3	\mathcal{C}_4	\mathcal{C}_5	\mathcal{C}_6
$D_1 [L_{168}]$	1	1	1	1	1	1
$D_6 [L_{168}]$	6	2	0	0	-1	-1
$D_7 [L_{168}]$	7	-1	1	-1	0	0
$D_8 [L_{168}]$	8	0	-1	0	1	1
$DA_3 [L_{168}]$	3	-1	0	1	$\frac{1}{2}(-1 + i\sqrt{7})$	$\frac{1}{2}(-1 - i\sqrt{7})$
$DB_3 [L_{168}]$	3	-1	0	1	$\frac{1}{2}(-1 - i\sqrt{7})$	$\frac{1}{2}(-1 + i\sqrt{7})$

$$(5.4)$$

For our purposes the most interesting representation is the 7 dimensional one. Indeed its properties are the very reason to consider the group L_{168} in the present context. The following three statements are true:

1. The 7-dimensional irreducible representation is crystallographic since all elements $\gamma \in L_{168}$ are represented by integer valued matrices $D_7(\gamma)$ in a basis of vectors that span a lattice, namely the root lattice Λ_{root} of the A_7 simple Lie algebra.
2. The 7-dimensional irreducible representation provides an immersion $L_{168} \hookrightarrow \text{SO}(7)$ since its elements preserve the symmetric Cartan matrix of A_7 :

$$\begin{aligned} \forall \gamma \in L_{168} \quad : \quad D_7^T(\gamma) \mathcal{C} D_7(\gamma) &= \mathcal{C} \\ \mathcal{C}_{i,j} &= \alpha_i \cdot \alpha_j \quad (i, j = 1 \dots, 7) \end{aligned} \quad (5.5)$$

defined in terms of the simple roots α_i whose standard construction in terms of the unit vectors ϵ_i of \mathbb{R}^8 is recalled below:

$$\begin{aligned}\alpha_1 &= \epsilon_1 - \epsilon_2 & ; & & \alpha_2 &= \epsilon_2 - \epsilon_3 & = & ; & \alpha_3 &= \epsilon_3 - \epsilon_4 \\ \alpha_4 &= \epsilon_4 - \epsilon_5 & ; & & \alpha_5 &= \epsilon_5 - \epsilon_6 & = & ; & \alpha_6 &= \epsilon_6 - \epsilon_7 \\ \alpha_7 &= \epsilon_7 - \epsilon_8\end{aligned}\tag{5.6}$$

3. Actually the 7-dimensional representation defines an embedding $L_{168} \hookrightarrow G_2 \subset SO(7)$ since there exists a three-index antisymmetric tensor ϕ_{ijk} satisfying the relations of octonionic structure constants that is preserved by all the matrices $D_7(\gamma)$:

$$\forall \gamma \in L_{168} \quad : \quad D_7(\gamma)_{ii'} D_7(\gamma)_{jj'} D_7(\gamma)_{kk'} \phi_{i'j'k'} = \phi_{ijk}\tag{5.7}$$

Let us prove the above statements. It suffices to write the explicit form of the generators R , S and T in the crystallographic basis of the considered root lattice:

$$\mathbf{v} \in \Lambda_{\text{root}} \quad \Leftrightarrow \quad \mathbf{v} = n_i \alpha_i \quad n_i \in \mathbb{Z}\tag{5.8}$$

Explicitly if we set:

$$\begin{aligned}\mathcal{R} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \quad \mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \mathcal{T} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}\tag{5.9}$$

we find that the defining relations of L_{168} are satisfied:

$$\mathcal{R}^2 = \mathcal{S}^3 = \mathcal{T}^7 = \mathcal{RST} = (\mathcal{TSR})^4 = \mathbf{1}_{7 \times 7}\tag{5.10}$$

and furthermore we have:

$$\mathcal{R}^T \mathcal{C} \mathcal{R} = \mathcal{S}^T \mathcal{C} \mathcal{S} = \mathcal{T}^T \mathcal{C} \mathcal{T} = \mathcal{C}\tag{5.11}$$

where the explicit form of the A_7 Cartan matrix is recalled below:

$$\mathcal{C} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \quad (5.12)$$

This proves statements 1) and 2).

In order to prove statement 3) we proceed as follows. In \mathbb{R}^7 we consider the antisymmetric three-index tensor ϕ_{ABC} that, in the standard orthonormal basis, has the following components:

$$\begin{aligned} \phi_{1,2,6} &= \frac{1}{6} \\ \phi_{1,3,4} &= -\frac{1}{6} \\ \phi_{1,5,7} &= -\frac{1}{6} \\ \phi_{2,3,7} &= \frac{1}{6} \quad ; \quad \text{all other components vanish} \\ \phi_{2,4,5} &= \frac{1}{6} \\ \phi_{3,5,6} &= -\frac{1}{6} \\ \phi_{4,6,7} &= -\frac{1}{6} \end{aligned} \quad (5.13)$$

This tensor satisfies the algebraic relations of octonionic structure constants, namely³:

$$\phi_{ABM} \phi_{CDM} = \frac{1}{18} \delta_{CD}^{AB} + \frac{2}{3} \Phi_{ABCD} \quad (5.14)$$

$$\phi_{ABC} = -\frac{1}{6} \epsilon_{ABCPQRS} \Phi_{ABCD} \quad (5.15)$$

and the subgroup of $\text{SO}(7)$ which leaves ϕ_{ABC} invariant is, by definition, the compact section $\text{G}_{(2,-14)}$ of the complex G_2 Lie group (see for instance [9]). A particular matrix that transforms the standard

³In this equation the indices of the G_2 -invariant tensor are denoted with capital letter of the Latin alphabet, as it was the case in the quoted literature on weak G_2 -structures. In the following we will use lower case latin letters to be consistent with our notation for the supergravity constructions, the upper Latin letters being reserved for $d = 8$

orthonormal basis of \mathbb{R}^7 into the basis of simple roots α_i is the following one:

$$\mathfrak{M} = \begin{pmatrix} \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \sqrt{2} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad (5.16)$$

since:

$$\mathfrak{M}^T \mathfrak{M} = \mathcal{C} \quad (5.17)$$

Defining the transformed tensor:

$$\varphi_{ijk} \equiv (\mathfrak{M}^{-1})_i^I (\mathfrak{M}^{-1})_j^J (\mathfrak{M}^{-1})_k^K \phi_{IJK} \quad (5.18)$$

we can explicitly verify that:

$$\begin{aligned} \varphi_{ijk} &= (\mathcal{R})_i^p (\mathcal{R})_j^q (\mathcal{R})_k^r \varphi_{pqr} \\ \varphi_{ijk} &= (\mathcal{S})_i^p (\mathcal{S})_j^q (\mathcal{S})_k^r \varphi_{pqr} \\ \varphi_{ijk} &= (\mathcal{T})_i^p (\mathcal{T})_j^q (\mathcal{T})_k^r \varphi_{pqr} \end{aligned} \quad (5.19)$$

Hence, being preserved by the three-generators \mathcal{R}, \mathcal{S} and \mathcal{T} , the antisymmetric tensor φ_{ijk} is preserved by the entire discrete group L_{168} which, henceforth, is a subgroup of $G_{(2,-14)} \subset SO(7)$, as it was shown by intrinsic group theoretical arguments in [4].

6 Classification of the proper subgroups $H \subset L_{168}$ and of the L_{168} -orbits in the weight lattice

We aim at the construction of solutions of Englert equation (2.31) on the crystallographic 7-torus:

$$T^7 \equiv \frac{\mathbb{R}^7}{\Lambda_{\text{root}}} \quad (6.1)$$

where the root lattice is defined in eq.(5.8). The first necessary step is to introduce the dual weight lattice

$$\Lambda_w \ni \mathbf{w} = n_i \lambda^i \quad : \quad n^i \in \mathbb{Z} \quad (6.2)$$

spanned by the simple weights that are implicitly defined by the relations:

$$\lambda^i \cdot \alpha_j = \delta_j^i \quad \Rightarrow \quad \lambda^i = (\mathcal{C}^{-1})^{ij} \alpha_j \quad (6.3)$$

Given the generators of the group L_{168} in the basis of simple roots we obtain the same in the basis of simple weights through the following transformation:

$$\mathcal{R}_w = \mathcal{C} \mathcal{R} \mathcal{C}^{-1} \quad ; \quad \mathcal{S}_w = \mathcal{C} \mathcal{S} \mathcal{C}^{-1} \quad ; \quad \mathcal{T}_w = \mathcal{C} \mathcal{T} \mathcal{C}^{-1} \quad (6.4)$$

Explicitly we find:

$$\mathcal{R}_w = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \mathcal{S}_w = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6.5)$$

$$\mathcal{T}_w = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (6.6)$$

Equipped with this result we can construct orbits of weight lattice vectors under the action of the group L_{168} .

6.1 A first random exploration of the orbits

As a first orientation exercise we resorted to random calculations and we found that there are orbits \mathcal{O} whose length $\ell_{\mathcal{O}}$ takes one among the following seven values:

$$\ell_{\mathcal{O}} \in \{168, 84, 56, 42, 28, 14, 8\} \quad (6.7)$$

In the next subsection we will retrieve these numbers from a rigorous and exhaustive classification of all conjugacy classes of the proper subgroups $H \subset L_{168}$. Indeed given a vector $\vec{v}_0 \in \Lambda_w$, the L_{168} -orbit $\mathcal{O}(\vec{v}_0)$ of such a vector is isomorphic to the coset:

$$L_{168}/H_0^s \quad (6.8)$$

where H_0^s is the stability subgroup of \vec{v}_0 , that is:

$$\forall \gamma \in H_0^s \quad : \quad \gamma \cdot \vec{v}_0 = \vec{v}_0 \quad (6.9)$$

Every other vector in the same orbit $\vec{v} \in \mathcal{O}(\vec{v}_0)$ has a stability subgroup $H^s \subset L_{168}$ which is conjugate to H_0 via the group element $g \in L_{168}$ which maps \vec{v}_0 into \vec{v} . Hence the classification of all the possible orbits amounts to the classification of the conjugacy classes of possible stability subgroups H^s which certainly is included in the classification of conjugacy classes of proper subgroups $H \subset L_{168}$. The latter sentence is a *caveat*. It might happen that a certain subgroup H admits no fixed vector $v \in \Lambda_w$. In that case there is no orbit with such a stability subgroup and the corresponding coset is not isomorphic to any orbit.

Before plunging into the above sketched systematics, let us note that some of the numbers in eq.(6.7) correspond to the dimensions, of certain irreducible representations of $SL(8, \mathbb{R})$, other instead do not correspond to the dimensions of any representation. In particular we have:

$$\dim \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = 168 \quad ; \quad \dim \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} = 56 \quad ; \quad \dim \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = 28 \quad ; \quad \dim \square = 8 \quad (6.10)$$

In the above cases we have verified that the set of weights of the corresponding representation coincides with the L_{168} -orbit of its maximal weight λ_{max} .

Yet, as it will clearly appear from the exhaustive discussion of the next section, this is just a curious coincidence but it is not the key to understand the complete classification of orbits.

6.2 Classification of conjugacy classes of subgroups $H \subset L_{168}$

An important mathematical result states that the simple group L_{168} contains maximal subgroups only of index 8 and 7, namely of order 21 and 24 [29]. The order 21 subgroup G_{21} is the unique non-abelian group of that order and abstractly it has the structure of the semidirect product $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$. Up to conjugation there is only one subgroup G_{21} as we have explicitly verified with the computer. On the other hand, up to conjugation, there are two different groups of order 24 that are both isomorphic to the octahedral group O_{24} .

6.2.1 The maximal subgroup G_{21}

The group G_{21} has two generators \mathcal{X} and \mathcal{Y} that satisfy the following relations:

$$\mathcal{X}^3 = \mathcal{Y}^7 = 1 \quad ; \quad \mathcal{X}\mathcal{Y} = \mathcal{Y}^2\mathcal{X} \quad (6.11)$$

The organization of the 21 group elements into conjugacy classes is displayed below:

ConjugacyClass	C_1	C_2	C_3	C_4	C_5
representative of the class	e	\mathcal{Y}	$\mathcal{X}^2\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	\mathcal{X}
order of the elements in the class	1	7	7	3	3
number of elements in the class	1	3	3	7	7

(6.12)

As we see there are five conjugacy classes which implies that there should be five irreducible representations the square of whose dimensions should sum up to the group order 21. The solution of this problem is:

$$21 = 1^2 + 1^2 + 1^2 + 3^2 + 3^2 \quad (6.13)$$

and the corresponding character table is mentioned below:

0	e	\mathcal{Y}	$\mathcal{X}^2\mathcal{Y}\mathcal{X}\mathcal{Y}^2$	$\mathcal{Y}\mathcal{X}^2$	\mathcal{X}
$D_1 [G_{21}]$	1	1	1	1	1
$DX_1 [G_{21}]$	1	1	1	$-(-1)^{1/3}$	$(-1)^{2/3}$
$DY_1 [G_{21}]$	1	1	1	$(-1)^{2/3}$	$-(-1)^{1/3}$
$DA_3 [G_{21}]$	3	$\frac{1}{2}i(i + \sqrt{7})$	$-\frac{1}{2}i(-i + \sqrt{7})$	0	0
$DB_3 [G_{21}]$	3	$-\frac{1}{2}i(-i + \sqrt{7})$	$\frac{1}{2}i(i + \sqrt{7})$	0	0

(6.14)

In the weight-basis the two generators of the G_{21} subgroup of L_{168} can be chosen to be the following matrices and this fixes our representative of the unique conjugacy class:

$$\mathcal{X} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \quad \mathcal{Y} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad (6.15)$$

6.2.2 The maximal subgroups O_{24A} and O_{24B}

The octahedral group O_{24} has two generators S and T that satisfy the following relations:

$$S^2 = T^3 = (ST)^4 = \mathbf{1} \quad (6.16)$$

The 24 elements are organized in five conjugacy classes according to the scheme displayed below:

Conjugacy Class	C_1	C_2	C_3	C_4	C_5
representative of the class	e	T	$STST$	S	ST
order of the elements in the class	1	3	2	2	4
number of elements in the class	1	8	3	6	6

(6.17)

It follows that there are five irreducible representations that turn out to be of dimensions, 1,1,2,3,3, according to the sum rule:

$$24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2 \quad (6.18)$$

The corresponding character table is the following one:

0	e	T	$STST$	S	ST
$D_1 [O_{24}]$	1	1	1	1	1
$DX_1 [O_{24}]$	1	1	1	-1	-1
$D_2 [O_{24}]$	2	-1	2	0	0
$DA_3 [O_{24}]$	3	0	-1	-1	1
$DB_3 [O_{24}]$	3	0	-1	1	-1

(6.19)

By computer calculations we have verified that there are just two disjoint conjugacy classes of O_{24} maximal subgroups in L_{168} that we have named A and B, respectively. We have chosen two standard representatives, one for each conjugacy class, that we have named O_{24A} and O_{24B} respectively. To fix these subgroups it suffices to mention the explicit form of the their generators in the weight basis.

For the group O_{24A} , we chose:

$$T_A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \end{pmatrix} \quad S_A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad (6.20)$$

For the group O_{24B} , we chose:

$$T_B = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad S_B = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (6.21)$$

6.2.3 The tetrahedral subgroup $T_{12} \subset O_{24}$

Every octahedral group O_{24} has, up to O_{24} -conjugation, a unique tetrahedral subgroup T_{12} whose order is 12. The abstract description of the tetrahedral group is provided by the following presentation in terms of two generators:

$$T_{12} = (s, t \mid s^2 = t^3 = (st)^3 = 1) \quad (6.22)$$

The 12 elements are organized into four conjugacy classes as displayed below:

Classes	C_1	C_2	C_3	C_4
standard representative	1	s	t	t^2s
order of the elements in the conjugacy class	1	2	3	3
number of elements in the conjugacy class	1	3	4	4

(6.23)

We do not display the character table since we will not use it in the present paper. We anticipate that the two tetrahedral subgroups $T_{12A} \subset O_{24A}$ and $T_{12B} \subset O_{24B}$ are not conjugate under the big group L_{168} . Hence we have two conjugacy classes of tetrahedral subgroups of L_{168} , as we explain in the sequel.

6.2.4 The dihedral subgroup $Dih_3 \subset O_{24}$

Every octahedral group O_{24} has a dihedral group Dih_3 whose order is 6. The abstract description of the dihedral group Dih_3 is provided by the following presentation in terms of two generators:

$$Dih_3 = (A, B \mid A^3 = B^2 = (BA)^2 = 1) \quad (6.24)$$

The 6 elements are organized into three conjugacy classes as displayed below:

ConjugacyClasses	C_1	C_2	C_3
standard representative of the class	1	A	B
order of the elements in the class	1	3	2
number of elements in the class	1	2	3

(6.25)

We do not display the character table since we will not use it in the present paper. We anticipate that differently from the case of the tetrahedral subgroups the two dihedral subgroups $Dih_{3A} \subset O_{24A}$ and $Dih_{3B} \subset O_{24B}$ turn out to be conjugate under the big group L_{168} . Actually there is just one L_{168} -conjugacy class of dihedral subgroups Dih_3 .

6.2.5 Enumeration of the possible subgroups and orbits

Since the maximal subgroups of L_{168} are of index 7 or 8 we can have subgroups $H \subset L_{168}$ that are either G_{21} or O_{24} or subgroups thereof. Furthermore, as it is well known, the order $|H|$ of any subgroup $H \subset G$ must be a divisor of $|G|$. Hence we conclude that

$$|H| \in \{1, 2, 3, 4, 6, 7, 8, 12, 21, 24\} \quad (6.26)$$

Correspondingly we might have L_{168} -orbits \mathcal{O} in the weight lattice Λ_w , whose length is one of the following nine numbers:

$$\ell_{\mathcal{O}} \in \{168, 84, 56, 42, 28, 24, 21, 14, 8, 7\} \quad (6.27)$$

Comparing eq.(6.27) with the result of our numerical random experiment displayed in eq.(6.7) we see that three orbit lengths are excluded, namely 24, 21 and 7. In the sequel of this subsection we will

show the reason for these exclusions.

Combining the information about the possible group orders (6.26) with the information that the maximal subgroups are of index 8 or 7, we arrive at the following list of possible subgroups H (up to conjugation) of the group L_{168} :

Order 24) Either $H = O_{24A}$ or $H = O_{24B}$.

Order 21) The only possibility is $H = G_{21}$.

Order 12) The only possibilities are $H = T_{12A}$ or $H = T_{12B}$ where T_{12} is the tetrahedral subgroup of the octahedral group O_{24} .

Order 8) Either $H = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $H = \mathbb{Z}_2 \times \mathbb{Z}_4$.

Order 7) The only possibility is \mathbb{Z}_7 .

Order 6) Either $H = \mathbb{Z}_2 \times \mathbb{Z}_3$ or $H = \text{Dih}_3$, where Dih_3 denotes the dihedral subgroup of index 3 of the octahedral group O_{24} .

Order 4) Either $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $H = \mathbb{Z}_4$.

Order 3) The only possibility is $H = \mathbb{Z}_3$.

Order 2) The only possibility is $H = \mathbb{Z}_2$.

6.2.6 Summary of our results for the subgroups and the orbits

Let us summarize the results that we have derived by means of computer aided calculations.

1. We have verified that there are no orbits with stability subgroups either O_{24A} or O_{24B} . Indeed the constraints imposed on a seven vector \mathbf{v} by the request that it should be an eigenstate of the generators (6.20) or (6.21) admits the only solution $\mathbf{v} = 0$. *This means that there are no orbits of length 7.*
2. On the contrary we have verified that there are orbits with stability subgroup G_{21} . These orbits have length $\ell_{\mathcal{O}} = 8$ and depend from a *unique integer parameter* n . Indeed the most general vector \mathbf{v}_0 invariant under G_{21} has the following form:

$$\mathbf{v}_0 = \{0, 0, 0, n, -n, 0, 0\} \tag{6.28}$$

and the corresponding L_{168} -orbit is displayed below:

$$\mathcal{O}_8 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 & n & -n \\ 0 & 0 & 0 & 0 & n & -n & 0 \\ 0 & 0 & 0 & n & -n & 0 & 0 \\ 0 & 0 & n & -n & 0 & 0 & 0 \\ 0 & n & -n & 0 & 0 & 0 & 0 \\ -n & 0 & 0 & 0 & 0 & 0 & 0 \\ n & -n & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \quad (6.29)$$

every line denoting the components of a 7-vector belonging to the orbit.

3. As we anticipated above, by means of computer calculations we have verified that there are *two conjugacy classes* of *tetrahedral groups*, T_{12A} and T_{12B} . This implies that there are *two types* of orbits of length 14, namely \mathcal{O}_{14A} and \mathcal{O}_{14B} . Both of them depend only on *one integer parameter* n .

- In the case of T_{12A} , the two generators s, t can be chosen as follows:

$$\begin{aligned} s_A &= \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 0 & 0 \end{pmatrix} \\ t_A &= \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix} \end{aligned} \quad (6.30)$$

the most general vector \mathbf{v}_0 invariant under T_{12A} has the following form:

$$\mathbf{v}_0 = \{n, -n, n, 0, -n, 0, n\} \quad (6.31)$$

and the corresponding L_{168} -orbit is displayed below:

$$\mathcal{O}_{14A} = \left\{ \begin{pmatrix} 0 & 0 & -n & n & -n & 0 & 0 \\ 0 & 0 & n & -n & n & 0 & 0 \\ 0 & -n & 0 & 0 & n & -n & n \\ 0 & -n & n & -n & 0 & n & -n \\ 0 & n & 0 & 0 & -n & n & -n \\ 0 & n & -n & n & 0 & -n & n \\ -n & 0 & 0 & n & -n & n & 0 \\ -n & 0 & n & 0 & 0 & -n & 0 \\ -n & n & 0 & -n & 0 & 0 & n \\ -n & n & -n & 0 & n & 0 & -n \\ n & 0 & 0 & -n & n & -n & 0 \\ n & 0 & -n & 0 & 0 & n & 0 \\ n & -n & 0 & n & 0 & 0 & -n \\ n & -n & n & 0 & -n & 0 & n \end{pmatrix} \right\} \quad (6.32)$$

- In the case of T_{12B} , the two generators s, t can be chosen as follows:

$$\begin{aligned} s_B &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \end{pmatrix} \\ t_B &= \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (6.33)$$

the most general vector \mathbf{v}_0 invariant under T_{12B} has the following form:

$$\mathbf{v}_0 = \{n, 0, -n, n, -n, 0, n\} \quad (6.34)$$

and the corresponding L_{168} -orbit is displayed below:

$$\mathcal{O}_{14B} = \begin{pmatrix} 0 & 0 & -n & 0 & n & -n & 0 \\ 0 & 0 & n & 0 & -n & n & 0 \\ 0 & -n & 0 & n & -n & n & -n \\ 0 & -n & n & -n & 0 & 0 & n \\ 0 & n & 0 & -n & n & -n & n \\ 0 & n & -n & n & 0 & 0 & -n \\ -n & 0 & 0 & n & 0 & -n & n \\ -n & 0 & n & -n & n & 0 & -n \\ -n & n & 0 & 0 & -n & 0 & 0 \\ -n & n & -n & 0 & 0 & n & 0 \\ n & 0 & 0 & -n & 0 & n & -n \\ n & 0 & -n & n & -n & 0 & n \\ n & -n & 0 & 0 & n & 0 & 0 \\ n & -n & n & 0 & 0 & -n & 0 \end{pmatrix} \quad (6.35)$$

4. Next, as we anticipated above, we have verified that there is only *one conjugacy class* of *dihedral groups* Dih_3 . This implies that there is *only one type* of orbits of length 28. They depend only *on one integer parameter* n . Indeed a choice of the two generators A, B introduced in eq.(6.24) is the following one:

$$A = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad (6.36)$$

and the vector \mathbf{v}_0 invariant under the group they generate is the following one:

$$\mathbf{v}_0 = \{n, -n, 0, 0, 0, n, -n\} \quad (6.37)$$

The corresponding L_{168} -orbit is displayed below

$$O_{28} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 & n & 0 & -n \\ 0 & 0 & 0 & 0 & n & -n & n \\ 0 & 0 & 0 & n & 0 & -n & 0 \\ 0 & 0 & 0 & n & -n & 0 & n \\ 0 & 0 & 0 & n & -n & n & -n \\ 0 & 0 & n & 0 & -n & 0 & 0 \\ 0 & 0 & n & -n & 0 & 0 & n \\ 0 & 0 & n & -n & 0 & n & -n \\ 0 & 0 & n & -n & n & -n & 0 \\ 0 & -n & 0 & 0 & 0 & 0 & 0 \\ 0 & n & 0 & -n & 0 & 0 & 0 \\ 0 & n & -n & 0 & 0 & 0 & n \\ 0 & n & -n & 0 & 0 & n & -n \\ 0 & n & -n & n & -n & 0 & 0 \\ -n & 0 & 0 & 0 & 0 & 0 & n \\ -n & 0 & 0 & 0 & 0 & n & -n \\ -n & 0 & 0 & 0 & n & -n & 0 \\ -n & 0 & 0 & n & -n & 0 & 0 \\ -n & 0 & n & -n & 0 & 0 & 0 \\ -n & n & -n & 0 & 0 & 0 & 0 \\ n & 0 & -n & 0 & 0 & 0 & 0 \\ n & -n & 0 & 0 & 0 & 0 & n \\ n & -n & 0 & 0 & 0 & n & -n \\ n & -n & 0 & 0 & n & -n & 0 \\ n & -n & 0 & n & -n & 0 & 0 \\ n & -n & n & -n & 0 & 0 & 0 \end{pmatrix} \right\} \quad (6.38)$$

5. Next we have verified that there are two conjugacy classes of groups $H = \mathbb{Z}_2 \times \mathbb{Z}_2$. Yet any vector invariant with respect to any $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ is automatically invariant with respect to the tetrahedral group T_{12} of which H is a subgroup. This implies that we can obtain orbits of length 42 only from a stability subgroup $H = \mathbb{Z}_4$. It also implies that there cannot be orbits of length 21 with stability subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
6. Proceeding further we have verified that there is a unique conjugacy class of \mathbb{Z}_4 subgroups. Hence there is a unique type of orbits of length 42. They depend on a single integer parameter n . Taking as generator of the \mathbb{Z}_4 group the following matrix:

$$g_{\mathbb{Z}_4} = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (6.39)$$

the vector invariant under the action of such a \mathbb{Z}_4 is the following one:

$$\mathbf{v}_0 = \{0, 0, n, 0, 0, 0, -n\} \quad (6.40)$$

and the corresponding L_{168} -orbit is displayed below:

$$O_{42} = \begin{pmatrix} 0 & 0 & 0 & -n & 0 & 0 & 0 \\ 0 & 0 & 0 & n & 0 & 0 & 0 \\ 0 & 0 & -n & 0 & 0 & 0 & n \\ 0 & 0 & -n & 0 & 0 & n & -n \\ 0 & 0 & n & 0 & 0 & 0 & -n \\ 0 & 0 & n & 0 & 0 & -n & n \\ 0 & -n & 0 & 0 & 0 & n & 0 \\ 0 & -n & 0 & 0 & n & 0 & -n \\ 0 & -n & 0 & n & 0 & -n & 0 \\ 0 & -n & 0 & n & -n & 0 & n \\ 0 & -n & n & 0 & -n & 0 & 0 \\ 0 & -n & n & -n & n & -n & 0 \\ 0 & n & 0 & 0 & 0 & -n & 0 \\ 0 & n & 0 & 0 & -n & 0 & n \\ 0 & n & 0 & -n & 0 & n & 0 \\ 0 & n & 0 & -n & n & 0 & -n \\ 0 & n & -n & 0 & n & 0 & 0 \\ 0 & n & -n & n & -n & n & 0 \\ -n & 0 & 0 & 0 & n & 0 & 0 \\ -n & 0 & 0 & n & 0 & 0 & -n \\ -n & 0 & n & 0 & -n & 0 & n \\ -n & 0 & n & 0 & -n & n & -n \\ -n & 0 & n & -n & 0 & n & 0 \\ -n & 0 & n & -n & n & -n & n \\ -n & n & 0 & -n & 0 & n & -n \\ -n & n & 0 & -n & n & -n & 0 \\ -n & n & -n & 0 & n & -n & n \\ -n & n & -n & n & 0 & -n & 0 \\ -n & n & -n & n & -n & 0 & n \\ -n & n & -n & n & -n & n & -n \\ n & 0 & 0 & 0 & -n & 0 & 0 \\ n & 0 & 0 & -n & 0 & 0 & n \\ n & 0 & -n & 0 & n & 0 & -n \\ n & 0 & -n & 0 & n & -n & n \\ n & 0 & -n & n & 0 & -n & 0 \\ n & 0 & -n & n & -n & n & -n \\ n & -n & 0 & n & 0 & -n & n \\ n & -n & 0 & n & -n & n & 0 \\ n & -n & n & 0 & -n & n & -n \\ n & -n & n & -n & 0 & n & 0 \\ n & -n & n & -n & n & 0 & -n \\ n & -n & n & -n & n & -n & n \end{pmatrix} \quad (6.41)$$

7. In the next step we have verified that there is a *unique conjugacy class of subgroups* \mathbb{Z}_3 . This

implies that there is a *unique type* of orbits of length 56. These depend on three integer parameters n, m, p . Indeed, taking as \mathbb{Z}_3 -generator the following element of L_{168} :

$$gz_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 \end{pmatrix} \quad (6.42)$$

the vector invariant under \mathbb{Z}_3 is the following one:

$$v_0 = \{0, 0, n, m, p, -m - p, 0\} \quad (6.43)$$

and the corresponding orbit is displayed in fig.1

8. Furthermore we verified that up to conjugation there is only one \mathbb{Z}_7 subgroup of L_{168} and that any vector \mathbf{v} that is invariant with respect to this \mathbb{Z}_7 is also invariant with respect to the G_{21} group which contains it. *Hence there are no orbits of length 24.*
9. Furthermore we verified that *there is no $\mathbb{Z}_2 \times \mathbb{Z}_4$ subgroup of O_{24} and hence of L_{168} . Hence orbits of length 21 do not exist.*
10. Finally, since in L_{168} there is a unique conjugacy class of elements of order 2, it follows that there is a unique conjugacy class of \mathbb{Z}_2 subgroups. Hence there is a unique type of orbits of length 84, depending on 3 integer parameters n, m, p . To this effect it suffices to take anyone of the 21 elements belonging to the second conjugacy class of L_{168} as \mathbb{Z}_2 -generator and the results follows. The orbit is shown in fig.2.

6.2.7 Synopsis of the L_{168} orbits in the weight lattice Λ_w

Our findings about the available L_{168} -orbits in the weight lattice are summarized below

1. Orbits of length 8 (one parameter \mathbf{n} ; stability subgroup $H^s = G_{21}$)
2. Orbits of length 14 (two types A & B) (one parameter \mathbf{n} ; stability subgroup $H^s = T_{12A,B}$)
3. Orbits of length 28 (one parameter \mathbf{n} ; stability subgroup $H^s = Dih_3$)
4. Orbits of length 42 (one parameter \mathbf{n} ; stability subgroup $H^s = \mathbb{Z}_4$))
5. Orbits of length 56 (three parameters $\mathbf{n, m, p}$; stability subgroup $H^s = \mathbb{Z}_3$)
6. Orbits of length 84 (three parameters $\mathbf{n, m, p}$; stability subgroup $H^s = \mathbb{Z}_2$)
7. Generic orbits of length 168 (seven parameters ; stability subgroup $H^s = \mathbf{1}$)

$$\text{Orbit } 56 = \begin{pmatrix} 0 & 0 & -n & m+n+p & -p & -m-n & 0 \\ 0 & 0 & n & m & p & -m-p & 0 \\ 0 & m & -m-n & 0 & n & -n & m+n+p \\ 0 & -n & n & -n & 0 & m+n+p & -p \\ 0 & -n & m+n & -m-n & m+n+p & -m-n-p & n \\ 0 & -n & m+n+p & -p & -m-n & n & -n \\ 0 & n & m & p & -m-p & -n & n \\ 0 & n & -n & n & 0 & m & p \\ 0 & n & m+p & -m-p & m & -m & -n \\ 0 & m+n & -m & -n & n & m+p & -m-p \\ 0 & m+p & -m-n-p & n & -n & m+n & -m-n \\ 0 & m+n+p & -m-p & 0 & -n & n & m \\ -m & 0 & m+p & -m-n-p & n & -n & 0 \\ -m & -n & 0 & n & m+p & -m-p & -n \\ -m & -n & n & m+p & -m-n-p & 0 & n \\ m & -m & -n & 0 & n & m+p & -m-n-p \\ m & -m-n & 0 & n & -n & n & m+p \\ m & p & -m-p & -n & 0 & 0 & n \\ -m-n & 0 & n & -n & n & 0 & m+p \\ -m-n & n & 0 & 0 & -n & m+n+p & -m-n-p \\ -m-n & m+n+p & -m-n-p & n & 0 & -n & n \\ -n & 0 & 0 & n & m & p & -m-p \\ -n & 0 & n & m+p & -m-p & m & -m-n \\ -n & 0 & m+n+p & -m-p & 0 & -n & m+n \\ -n & n & 0 & m & -m-n & 0 & m+n+p \\ -n & n & -n & 0 & m+n+p & -m-p & m \\ -n & n & m+p & -m-n-p & 0 & m+n & -m \\ -n & m+n & -m & 0 & m+p & -m-n-p & 0 \\ -n & m+n & -m-n & m+n+p & -m-n-p & n & 0 \\ -n & m+n+p & -p & -m-n & n & 0 & -n \\ n & 0 & 0 & -n & m+n+p & -p & -m-n \\ n & 0 & m & -m-n & 0 & n & m+p \\ n & 0 & -n & m+n & -m-n & m+n+p & -m-p \\ n & m & p & -m-p & -n & 0 & n \\ n & -n & 0 & m+n+p & -m-p & 0 & m \\ n & -n & n & 0 & m & -m-n & m+n+p \\ n & -n & m+n & -m & 0 & m+p & -m-n-p \\ n & m+p & -m-p & m & -m & -n & 0 \\ n & m+p & -m-n-p & 0 & m+n & -m & 0 \\ m+n & -m & 0 & m+p & -m-n-p & n & -n \\ m+n & -m & -n & n & m+p & -m-n-p & n \\ m+n & -m-n & m+n+p & -m-n-p & n & 0 & 0 \\ -m-p & 0 & -n & n & -n & 0 & m+n \\ -m-p & m & -m & -n & 0 & n & -n \\ -m-p & -n & 0 & 0 & n & m & -m \\ -m-n-p & 0 & m+n & -m & -n & n & 0 \\ -m-n-p & n & 0 & -n & m+n & -m-n & n \\ -m-n-p & n & -n & m+n & -m & 0 & -n \\ -p & -m-n & n & 0 & 0 & -n & 0 \\ p & -m-p & -n & 0 & 0 & n & 0 \\ m+p & -m-p & m & -m & -n & 0 & 0 \\ m+p & -m-n-p & 0 & m+n & -m & -n & n \\ m+p & -m-n-p & n & -n & m+n & -m & -n \\ m+n+p & -m-p & 0 & -n & n & -n & m+n \\ m+n+p & -m-n-p & n & 0 & -n & m+n & -m \\ m+n+p & -p & -m-n & n & 0 & 0 & -n \end{pmatrix}$$

Figure 1: The vectors belonging to the length 56 orbit of L_{162} in the A_7 weight lattice

7 Solutions of Englert Equation associated with L_{168} orbits in the weight lattice

Let us now generalize the solution algorithm introduced in [19] for Beltrami equation on the crystallographic 3-torus:

$$T^3 \equiv \frac{\mathbb{R}^3}{\Lambda_{\text{cubic}}} \quad (7.1)$$

to the case of Englert equation (2.31) on the crystallographic 7-torus eq.(6.1). To this effect, let us choose as line element on the crystallographic torus (6.1) the following one:

$$ds_{T^7}^2 = \mathcal{C}_{ij} dX^i \otimes dX^j \quad (7.2)$$

0	-m	-n	n	m+p	0	-m-n-p
0	m	p	-p	-m-n	0	m+n+p
0	-m-n	n	m+p	-m-n-p	n	m+p
0	-n	0	m+n+p	-p	p	-p
0	n	0	m	p	-p	p
0	m+n	p	-m-p	m	p	-m-p
0	-m-p	-n	m+n	-m	-n	m+n
0	-m-n-p	n	-n	m+n	0	-m
0	-p	0	-m	-n	n	-n
0	p	0	-m-n-p	n	-n	n
0	m+p	-p	-m-n	m+n+p	-p	-m-n
0	m+n+p	-p	p	-m-p	0	m
-m	0	m+p	-p	-m-n	m+n+p	-m-n-p
-m	m	p	-m-p	m+p	-m-n-p	0
-m	-n	n	m+p	0	-p	-m-n
-m	-n	m+n	-m-n	m+n+p	-m-p	m+p
-m	-n	m+n+p	0	-m-p	-n	m+n
-m	m+p	-m-n-p	m+n+p	-p	-m-n	n
-m	0	-m-n	n	m+p	-m-n-p	m+n+p
-m	-m	-n	m+n	-m-n	m+n+p	0
-m	-m-n	m+n+p	-m-n-p	n	m+p	-p
-m	p	-m-p	m+p	-m-n-p	m+n	-m-n
-m	p	-m-n-p	0	m+n	0	-m-p
-m	p	-p	-m-n	0	n	m+p
-m-n	0	n	0	m	p	0
-m-n	n	m+p	-m-n-p	n	m	p
-m-n	m+n	-m	m+p	-m-n-p	m+n+p	-m-p
-m-n	m+n+p	-m-p	m	-m	-n	m+n+p
-m-n	m+n+p	-m-n-p	n	m+p	-m-p	m
-m-n	m+n+p	-p	-m	0	m+p	-m-n-p
-n	0	m+n+p	-p	p	-m-p	0
-n	n	m+p	0	-p	0	-m-n
-n	m+n	0	p	0	-m-n-p	n
-n	m+n	-m	-n	m+n+p	0	-p
-n	m+n	-m-n	m+n+p	-m-p	m	p
-n	m+n+p	0	-m-p	-n	m+n	0
n	0	m	p	-p	-m-n	m+n+p
n	m	0	-m-n	n	m+p	0
n	-n	m+n	0	p	0	-m-p
n	m+p	0	-p	0	-m	-n
n	m+p	-m-p	m	-m-n	m+n-p	-p
n	m+p	-m-n-p	n	m	0	p
m+n	0	p	0	-m-n-p	n	0
m+n	-m	m	p	-m-p	m+p	-m-n-p
m+n	-m	-n	m+n+p	0	-m-p	m
m+n	-m	m+p	-m-n-p	m+n+p	-m	-p
m+n	-m-n	m+n+p	-m-p	m	-m	m+p
m+n	p	-m-p	m	p	-m-n-p	n
-m-p	0	-n	0	m+n+p	-p	0
-m-p	m	-m	-n	m+n	-m-n	m+n+p
-m-p	m	-m-n	m+n+p	-m-n-p	n	m
-m-p	m	p	-m-n-p	0	m+n	-m
-m-p	-n	m+n	-m	-n	m+n+p	-p
-m-p	m+p	-m-n-p	m+n	-m	m	-m-n
-m-n-p	0	m+n	p	-m-p	m	-m
-m-n-p	n	m	0	-m-n	n	m+p
-m-n-p	n	-n	m+n	0	p	-m-p
-m-n-p	n	m+p	-m-p	m	-m-n	m+n
-m-n-p	m+n	-m	m	p	-m-p	-n
-m-n-p	m+n+p	-p	-m-n	m+n	-m	0
-p	0	-m	-n	n	m+p	-m-n-p
-p	-m	0	m+p	-p	-m-n	0
-p	-m-n	0	n	0	m	p
-p	-m-n	m+n	-m	m+p	-m-n-p	n
-p	-m-n	m+n+p	-p	-m	0	-n
-p	p	-m-p	0	-n	0	m+n
p	0	-m-n-p	n	-n	m+n	-m
p	-m-p	0	-n	0	m+n+p	-p
p	-m-p	m	p	-m-n-p	0	n
p	-m-p	m+p	-m-n-p	m+n	-m	-n
p	-m-n-p	0	m+n	p	-m-p	0
p	-p	-m-n	0	n	0	m+p
m+p	0	-p	0	-m	-n	0
m+p	-m-p	m	-m-n	m+n+p	-m-n-p	m+n
m+p	-m-n-p	n	m	0	-m-n	m+n+p
m+p	-m-n-p	m+n	-m	m	p	-m-n-p
m+p	-m-n-p	m+n+p	-p	-m-n	m+n	-m
m+p	-p	-m-n	m+n+p	-p	-m	-n
m+n+p	0	-m-p	-n	m+n	-m	m
m+n+p	-m-p	m	-m	-n	m+n	p
m+n+p	-m-n-p	n	m+p	-m-p	m	0
m+n+p	-p	-m	0	m+p	-p	-m-n
m+n+p	p	-m-n	m+n	-m	m+p	-m-p
m+n+p	-p	p	-m-p	0	-n	m+n

Figure 2: The vectors belonging to the length 84 orbit of L_{162} in the A_7 weight lattice

where \mathcal{C}_{ij} is the Cartan matrix (5.12). In relation with the notations of section 2.2.1 we have set:

$$x^i = \alpha_j^i X^j \quad \Rightarrow \quad x = \mathfrak{M} X \quad (7.3)$$

where α_j^i denotes the i -th component, in an orthonormal basis, of the j -th simple root, an explicit realization of which is indeed provided by the corresponding entry of the matrix \mathfrak{M} introduced in equation (5.16).

Secondly let us introduce the following ansatz:

$$\begin{aligned} \mathbf{Y}^{[3]} &= \sum_{\mathbf{k} \in \mathcal{O}} (v_{ijk}(\mathbf{k}) \cos[2\pi \mathbf{k} \cdot \mathbf{X}] + \omega_{ijk}(\mathbf{k}) \sin[2\pi \mathbf{k} \cdot \mathbf{X}]) dX^i \wedge dX^j \wedge dX^k \\ &\equiv \mathcal{Y}_{ijk}(\mathbf{X}) dX^i \wedge dX^j \wedge dX^k = Y_{ijk}(\mathbf{x}) dx^i \wedge dx^j \wedge dx^k \end{aligned} \quad (7.4)$$

where \mathcal{O} denotes some L_{168} orbit of momentum vectors in the weight lattice Λ_{weight} and where:

$$\mathbf{k} \cdot \mathbf{X} \equiv k_\ell X^\ell \quad (7.5)$$

Eq. (7.4) defines also the relation between the tensors $Y_{ijk}(\mathbf{x})$ and $\mathcal{Y}_{ijk}(\mathbf{X})$, namely:

$$Y_{ijk} = (\mathfrak{M}^{-1})_i^{i'} (\mathfrak{M}^{-1})_j^{j'} (\mathfrak{M}^{-1})_k^{k'} \mathcal{Y}_{i'j'k'} \quad (7.6)$$

With such an ansatz, Englert equation (2.31) is turned into the following pair of algebraic equations for the coefficients:

$$\sqrt{\det \mathcal{C}} \epsilon_{ijk}^{\ell mnp} k_\ell v_{mnp} = -\frac{6\mu}{\pi} \omega_{ijk} \quad (7.7)$$

$$\sqrt{\det \mathcal{C}} \epsilon_{ijk}^{\ell mnp} k_\ell \omega_{mnp} = \frac{6\mu}{\pi} v_{ijk} \quad (7.8)$$

where four indices of the Levi-Civita epsilon symbol have being raised with the inverse metric \mathcal{C}^{-1} . Substituting the first equation into the second we obtain the following consistency conditions:

$$\mu^2 = \pi^2 \|\mathbf{k}\|^2 \quad ; \quad \|\mathbf{k}\|^2 = k_\ell k_m (\mathcal{C}^{-1})^{\ell m} \quad (7.9)$$

$$0 = (\mathcal{C}^{-1})^{\ell m} k_\ell v_{ijm} = (\mathcal{C}^{-1})^{\ell m} k_\ell \omega_{ijm} \quad (7.10)$$

It is easy to count the number of parameters in the general solution of (7.8). There are a priori $35 + 35 = 70$ parameters for each momentum \mathbf{k} . Equation (7.10) imposes $21 + 21 = 42$ constraints. These latter are not all independent since the $7 + 7$ equations:

$$(\mathcal{C}^{-1})^{\ell m} k_\ell (\mathcal{C}^{-1})^{rn} k_r v_{inm} = (\mathcal{C}^{-1})^{\ell m} k_\ell (\mathcal{C}^{-1})^{rn} k_r \omega_{inm} = 0 \quad (7.11)$$

are automatically satisfied because of antisymmetry of the involved three-tensors. On their turn $1 + 1$ of the above equations follow from antisymmetry contracting once more with the momentum vector. In conclusion the number of independent parameters for each momentum vector is reduced by eq.(7.10) to

$$(35 - 21 + 7 - 1) \oplus (35 - 21 + 7 - 1) = 20 \oplus 20 \quad (7.12)$$

Finally eq.(7.8) halves this number so that for each momentum vector we have:

$$\# \text{ parameters} = 20 \quad (7.13)$$

We conclude that the number of parameters in a solution of Englert equation based on an orbit \mathcal{O} is

$$np = 20 \times |\mathcal{O}| \quad (7.14)$$

where $|\mathcal{O}|$ denotes the number of different weights contained in the considered orbit, counting weights that differ by an overall sign only once. The last specification is essential since cosine and sine are, respectively, an even and an odd function and we should not count $\cos[\pm 2\pi \mathbf{k} \cdot \mathbf{X}]$ and $\sin[\pm 2\pi \mathbf{k} \cdot \mathbf{X}]$ twice.

In this way from each orbit \mathcal{O} we obtain a 3-form

$$\mathbf{Y}_{\mathcal{O}}^{[3]}(\mathbf{X}|\mathbf{F}) \quad (7.15)$$

which satisfies Englert equation and depends on a set of np -parameters F_1, \dots, F_{np} collectively denoted \mathbf{F} .

The action of the crystallographic group L_{168} can be easily transferred from the torus coordinates to the parameter space solving the following linear equations:

$$\forall \gamma \in L_{168} \quad : \quad \mathbf{Y}_{\mathcal{O}}^{[3]}(D_7(\gamma) \cdot \mathbf{X}|\mathbf{F}) = \mathbf{Y}_{\mathcal{O}}^{[3]}(\mathbf{X}|\mathfrak{D}_{np}(\gamma) \cdot \mathbf{F}) \quad (7.16)$$

for the np^2 entries of the matrices $\mathfrak{D}_{np}(\gamma)$. In this way one constructs a np -dimensional linear representation of the group L_{168} which can always be decomposed into irreps using the character table in eq.(5.4).

8 A L_{168} -invariant Englert 3-form from the orbit \mathcal{O}_8

As a first exemplification of the procedure let us apply the described algorithm to the case of the shortest orbit \mathcal{O}_8 displayed in eq.(6.29). Utilizing a Mathematica code we produced the object:

$$\mathbf{Y}_{\mathcal{O}_8}^{[3]}(\mathbf{X}|\mathbf{F}_{160}) \quad (8.1)$$

depending on 160 real parameters and we easily constructed the 160×160 matrices of the reducible representation \mathfrak{D}_{160} . Then we calculated the traces of these matrices in each of the conjugation classes ordered as in table 5.3 and we obtained the following character:

$$\chi \equiv \chi[\mathfrak{D}_{160}] = \{160, 0, 4, 0, -1, -1\} \quad (8.2)$$

The multiplicities of the 6 irreducible representation follow immediately from the general formula:

$$a_{\mu} = \langle \chi, \chi_{\mu} \rangle \quad (8.3)$$

where the scalar product in character space is defined as follows:

$$\begin{aligned} \langle \chi, \psi \rangle &\equiv \frac{1}{168} \sum_{i=1}^6 g_i \chi^i \psi^i = \chi^i \kappa_{ij} \psi^j \\ \kappa &= \begin{pmatrix} \frac{1}{168} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{7} \end{pmatrix} \end{aligned} \quad (8.4)$$

the number g_i denoting the length of the conjugacy class \mathcal{C}_i . The result is the following multiplicity vector:

$$a_\mu = \{2, 6, 8, 6, 3, 3\} \quad (8.5)$$

which corresponds to the following decomposition into irreps:

$$\mathfrak{D}_{160} = 2 \mathbf{D}_1 \oplus 6 \mathbf{D}_6 \oplus 8 \mathbf{D}_7 \oplus 6 \mathbf{D}_8 \oplus 3 \mathbf{D}_{3a} \oplus 3 \mathbf{D}_{3b} \quad (8.6)$$

We conclude that there exists a 2-parameter solution of Englert equation, which is invariant under the full group L_{168} . It corresponds to the projection onto the \mathbf{D}_1 representation in eq.(8.6).

The projectors onto irreducible \mathbf{D}_μ can be obtained by means of another classical formula of finite group theory:

$$\mathbb{P}^\mu = \frac{1}{168} \sum_{i=1}^6 \chi_i^\mu \sum_{\gamma \in \mathcal{C}_i} \mathfrak{D}_{160}(\gamma) \quad (8.7)$$

Applying $\mathbb{P}^{[D_1]}$ to the parameter vector \mathbf{F}_{160} we set to zero 158 linear independent combinations of the F_i and the result is a 2-parameter three-form. We have explicitly verified that it is invariant under the full group L_{168} .

In order to display the explicit form of this solution we introduce the following set of 16 linearly independent trigonometric functions

$$\mathfrak{f}_\alpha(\mathbf{X}) = \begin{pmatrix} \cos(2\pi X_1) \\ \cos(2\pi(X_2 - X_1)) \\ \cos(2\pi(X_3 - X_2)) \\ \cos(2\pi(X_4 - X_3)) \\ \cos(2\pi(X_5 - X_4)) \\ \cos(2\pi(X_6 - X_5)) \\ \cos(2\pi X_7) \\ \cos(2\pi(X_7 - X_6)) \\ \sin(2\pi X_1) \\ \sin(2\pi(X_2 - X_1)) \\ \sin(2\pi(X_3 - X_2)) \\ \sin(2\pi(X_4 - X_3)) \\ \sin(2\pi(X_5 - X_4)) \\ \sin(2\pi(X_6 - X_5)) \\ -\sin(2\pi X_7) \\ \sin(2\pi(X_7 - X_6)) \end{pmatrix} ; \quad \alpha = 1, \dots, 16 \quad (8.8)$$

and we organize the 35 independent differentials of the integral coordinates into lexicographic order:

$$\Delta = \{dX_1 \wedge dX_2 \wedge dX_3, dX_1 \wedge dX_2 \wedge dX_4, \dots, dX_5 \wedge dX_6 \wedge dX_7\} = \{\Delta_q\}, \quad (q = 1, \dots, 35) \quad (8.9)$$

Then the L_{168} -invariant Englert 3-form can be written as follows

$$\mathbf{Y}^{[168]}(\mathbf{X}|\mathbf{F}) = \sum_{\alpha=1}^{16} \sum_{q=1}^{35} \mathfrak{C}_{q\alpha}^{[168]}(\hat{\mathbf{F}}) \Delta_q \times \mathfrak{f}_\alpha(\mathbf{X}) \quad (8.10)$$

where

$$\hat{\mathbf{F}} = \{F_1, F_2\} \quad (8.11)$$

and the coefficients $\mathfrak{C}_{q\alpha}^{[168]}(\hat{\mathbf{F}})$ are displayed in appendix A.

We have carefully studied the \mathcal{B} -operator defined in eq.(3.31) when it is polarized on $\mathbf{Y}^{[168]}(\mathbf{X}|\mathbf{F})$. To this effect one has to convert $\mathbf{Y}^{[168]}(\mathbf{X}|\mathbf{F})$ to the orthonormal coordinates and then use its redefined components in eq.(3.31). We have verified that for no choice of the parameter (F_1, F_2) , the rank of $\mathcal{B}^{[168]}$ can be reduced to be smaller than 8. Hence at least from the orbit \mathcal{O}_8 no solution emerges of Englert equation that is L_{168} -invariant and admits some residual supersymmetry.

9 Construction of a G_{21} -invariant solution of Englert equation

Motivated by the results of the previous section we have made a new searching for an Englert solution with $\mathcal{N} = 1$ supersymmetry and a reduced discrete symmetry $H \subset L_{168}$.

An educated guess suggests that we should rather use the maximal subgroup G_{21} . Indeed this latter contains a \mathbb{Z}_7 subgroup and no lattice of dimension less than seven can have a crystallographic realization of \mathbb{Z}_7 . This encourages to think that any solution of Englert equation invariant under G_{21} is intrinsically 7-dimensional. Since we could not find an L_{168} -invariant solution with $\mathcal{N} = 1$ supersymmetry, the next obvious possibility is to try with the maximal subgroup G_{21} .

It turns out that our guess is correct and in this section we construct a G_{21} -invariant solution of Englert equation which admits $\mathcal{N} = 1$ SUSY.

9.1 The G_{21} -orbit of length 7 in the weight lattice and its associated solution of Englert equation

The orbit $\mathcal{O}_7 \subset \Lambda_w$ of the group G_{21} in the weight lattice that we consider is made of 7 vectors and it is displayed below:

$$\mathcal{O}_7 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (9.1)$$

Using this orbit in the general construction algorithm (see eq.(7.4) and following ones) we construct a solution of Englert equation (2.33) with eigenvalue:

$$\mu = \frac{1}{2} \sqrt{\frac{7}{2}} \pi \quad (9.2)$$

By computer calculation we find a solution solution that depends on a set of 140 parameters:

$$\mathbf{F} = \{F_1, \dots, F_{140}\} \quad ; \quad F_A \quad (A = 1, \dots, 140) \quad (9.3)$$

and on a set of 14 independent trigonometric functions:

$$f_\alpha(\mathbf{X}) = \begin{pmatrix} \text{Cos}[2\pi X_1] \\ \text{Cos}[2\pi(-X_1 + X_2)] \\ \text{Cos}[2\pi(-X_2 + X_3)] \\ \text{Cos}[2\pi(-X_3 + X_4)] \\ \text{Cos}[2\pi(-X_5 + X_6)] \\ \text{Cos}[2\pi X_7] \\ \text{Cos}[2\pi(-X_6 + X_7)] \\ \text{Sin}[2\pi X_1] \\ \text{Sin}[2\pi(-X_1 + X_2)] \\ \text{Sin}[2\pi(-X_2 + X_3)] \\ \text{Sin}[2\pi(-X_3 + X_4)] \\ \text{Sin}[2\pi(-X_5 + X_6)] \\ -\text{Sin}[2\pi X_7] \\ \text{Sin}[2\pi(-X_6 + X_7)] \end{pmatrix} \quad (9.4)$$

This solution of Englert equation can be written as follows:

$$\mathbf{Y}(\mathbf{X}|\mathbf{F}) = \sum_{\alpha=1}^{14} \sum_{q=1}^{35} \mathfrak{C}_{q\alpha}(\mathbf{F}) \Delta_q \times f_\alpha(\mathbf{X}) \quad (9.5)$$

where the differentials Δ_q were defined in eq.(8.9) and where the $35 \times 14 = 490$ coefficients $\mathfrak{C}_{q\alpha}(\mathbf{F})$ are linear combinations of the F_A which we do not display for obvious reasons of space, since they are very large expressions.

Next we have derived the 140-dimensional representation \mathfrak{D}_{140} of the group $G_{21} \subset L_{168}$ induced on the parameter space by the standard identity (see eq.(7.16)):

$$\forall \gamma \in G_{21} \quad : \quad \mathbf{Y}(\gamma \mathbf{X}|\mathbf{F}) = \mathbf{Y}(\mathbf{X}|\mathfrak{D}_{140}[\gamma]\mathbf{F}) \quad (9.6)$$

The character of this representation turns out to be:

$$\chi^{[140]} = \{140, 0, 0, 2, 2\} \quad (9.7)$$

and by applying to \mathfrak{D}_{140} the standard group-theoretical formula that provides the multiplicity of any irrep in a given reducible representation

$$a^\mu = \frac{1}{21} \sum_{i=1}^5 g_i \bar{\chi}_i^\mu \chi_i^{[140]} \quad (9.8)$$

using for $\bar{\chi}_i^\mu$ the character table in eq.(6.14), we found the following decomposition of \mathfrak{D}_{140} into irreducible irreps of G_{21} :

$$\mathfrak{D}_{140} [G_{21}] = 8 D_1 [G_{21}] \oplus 20 DA_3 [G_{21}] \oplus 20 DB_3 [G_{21}] \oplus 6 DX_1 [G_{21}] \oplus 6 DY_1 [G_{21}] \quad (9.9)$$

the notations being those of eq.(6.14).

This means that from our constructed solution $\mathbf{Y}(\mathbf{X}|\mathbf{F})$ we can extract 8 singlets, namely an Englert solution, invariant under G_{21} which depends on 8 parameters. This solution can be easily obtained utilizing the group-theoretical projection onto any irreducible representation D^μ encoded into the following formula:

$$\mathbb{P}_{140}^{[\mu]} = \frac{1}{21} \sum_{i=1}^5 \chi_i^\mu \sum_{\gamma \in \mathcal{C}_i} \mathfrak{D}_{140}[\gamma] \quad (9.10)$$

where χ_i^μ is the i -th component of the μ -character and \mathcal{C}_i is the i -th conjugacy class of group elements. The 132 dimensional null-space of $\mathbb{P}_{140}^{[D_1]}$ provides us with 132 linear constraints on the parameters F_A that can be solved in terms of 8 parameters. We named these parameters as follows:

$$\Psi = \{\psi_1, \dots, \psi_8\} \quad (9.11)$$

The substitution of the solution of these constraints into eq.(9.5) produces a 3-form $\mathbf{Y}_s(\mathbf{X}|\Psi)$ where the subscript s stands for singlet (with respect to G_{21}).

9.2 Conversion of the G_{21} -invariant solution obtained in the root basis to the orthonormal basis of coordinates

The next step in our construction is the transcription of $\mathbf{Y}_s(\mathbf{X}|\Psi)$ that was constructed in the integral coordinates \mathbf{X} , adapted to the A7-root lattice, to the orthonormal coordinates \mathbf{x} , best suited to the construction of supergravity solutions. Applying eq.(7.3) we obtain:

$$\hat{\mathbf{Y}}_s(\mathbf{x}|\Psi) \equiv \mathbf{Y}_s(\mathfrak{M}^{-1}\mathbf{x}|\Psi) \quad (9.12)$$

and we can write a formula analogous to eq.(9.5)

$$\hat{\mathbf{Y}}_s(\mathbf{x}|\Psi) = \sum_{q=1}^{35} \sum_{\alpha=1}^{14} \mathfrak{C}_{q\alpha}(\Psi) \mathbf{d}_q \times \hat{f}_\alpha(\mathbf{x}) \quad (9.13)$$

where \mathbf{d}_q denote the lexicographic ordered differentials in the orthonormal basis:

$$\mathbf{d} = \{dx_1 \wedge dx_2 \wedge dx_3, dx_1 \wedge dx_2 \wedge dx_4, \dots, dx_5 \wedge dx_6 \wedge dx_7\} = \{\mathbf{d}_q\} \quad (q = 1, \dots, 35) \quad (9.14)$$

and $\hat{f}_\alpha(\mathbf{x})$ are the basis trigonometric functions in the same coordinates:

$$\hat{f}_\alpha(\mathbf{x}) = \begin{pmatrix} \cos \left[2\pi \left(\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}} \right) \right] \\ \cos \left[2\pi \left(-\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}} \right) \right] \\ \cos \left[2\pi \left(\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}} \right) \right] \\ \cos \left[2\pi \left(-\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{3x_6}{2\sqrt{2}} - \sqrt{2}x_6 - \frac{x_7}{2\sqrt{2}} \right) \right] \\ \cos \left[2\pi \left(-\frac{x_3}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}} \right) \right] \\ \cos \left[2\pi \left(\frac{x_4}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}} \right) \right] \\ \cos \left[2\pi \left(\frac{x_4}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}} \right) \right] \\ \sin \left[2\pi \left(\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}} \right) \right] \\ \sin \left[2\pi \left(-\frac{x_1}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}} \right) \right] \\ \sin \left[2\pi \left(\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}} \right) \right] \\ \sin \left[2\pi \left(-\frac{x_2}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{3x_6}{2\sqrt{2}} - \sqrt{2}x_6 - \frac{x_7}{2\sqrt{2}} \right) \right] \\ \sin \left[2\pi \left(-\frac{x_3}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}} \right) \right] \\ -\sin \left[2\pi \left(\frac{x_4}{\sqrt{2}} - \frac{x_5}{2\sqrt{2}} - \frac{x_6}{2\sqrt{2}} - \frac{x_7}{2\sqrt{2}} \right) \right] \\ \sin \left[2\pi \left(\frac{x_4}{\sqrt{2}} + \frac{x_5}{2\sqrt{2}} + \frac{x_6}{2\sqrt{2}} + \frac{x_7}{2\sqrt{2}} \right) \right] \end{pmatrix} ; \quad \alpha = 1, \dots, 14 \quad (9.15)$$

The full structure of the general solution is encoded in the coefficients $\hat{\mathbf{c}}_{q\alpha}(\Psi)$ which once again we do not display for reasons of space, being they very large objects.

The components $Y_{ijk}^s(\mathbf{x}|\Psi)$ of $\hat{\mathbf{Y}}_s(\mathbf{x}|\Psi)$ satisfy Englert equation in the orthonormal basis (see eq.(2.31)) with the appropriate value of μ , namely

$$\mu = -4\sqrt{\frac{7}{2}}\pi \quad (9.16)$$

and they can be used to construct an exact M2-brane solution of d=11 supergravity depending on the 8 moduli ψ_i .

9.3 Analysis of the \mathcal{B} -operator polarized on this solution

Our aim is that of finding, if possible, a subspace of the 8-dimensional moduli space where the rank of the 8×8 matrix:

$$\mathcal{B}(\mathbf{x}|\Psi) = \sum_{ijk} Y_{ijk}^s(\mathbf{x}|\Psi) \tau_{ijk} \quad (9.17)$$

is reduced, leading to the existence of some preserved supersymmetry. The strategy we have adopted for this task is the following. We have expanded the $\mathcal{B}(\mathbf{x}|\Psi)$ operator along the basis functions:

$$\mathcal{B}(\mathbf{x}|\Psi) = \sum_{\alpha=1}^{14} \mathcal{Q}_{\alpha}(\Psi) \hat{f}_{\alpha}(\mathbf{x}) \quad (9.18)$$

obtaining a set of 14 matrices $\mathcal{Q}_{\alpha}(\Psi)$ whose rank, for generic Ψ , is 8 for all of them. Next, in order to reduce the rank we have considered the condition that one of the rows (always the same) should be simultaneously zero for all the 14 matrices. Such conditions leads to a set of constraints with no solutions for all the rows except for the last one, namely the eight row. This row can be simultaneously annihilated for all the 14 matrices by a constraint that has a solution in terms of 4 parameters $(\psi_1, \psi_2, \psi_3, \psi_5)$, namely:

$$\begin{aligned} \psi_4 &= -3\psi_1 - 4\psi_2 - \psi_3 - 2\psi_5 \\ \psi_6 &= -2\psi_2 - 2\psi_5 \\ \psi_7 &= -5\psi_1 + 3\psi_2 - 4\psi_3 + 5\psi_5 \\ \psi_8 &= 5\psi_1 + \psi_2 + 3\psi_3 - \psi_5 \end{aligned} \quad (9.19)$$

Under these conditions not only the 8-th row of the operator $\mathcal{B}(\mathbf{x}|\Psi)$ vanishes but also so it does its 8-th column. In other words $\mathcal{B}(\mathbf{x}|\Psi)$ consistently reduces to a 7-dimensional operator in spinor space. Consequently the spinor:

$$\zeta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \zeta_8 \end{pmatrix} \quad (9.20)$$

introduced in the tensor product formula (4.11) leads to a Killing spinor of the considered 4-parameter M2-brane solution of d=11 supergravity.

Next, using the same procedure, we investigated whether the remaining 7×7 block of the \mathcal{B} -operator could be further reduced in rank. The answer was negative. Indeed it turns out that no condition can be imposed on the remaining 4-moduli ψ_i for which additional Killing spinors pop up. The conclusion is that our construction leads to an intrinsically $\mathcal{N} = 1$ M2-brane solution of d=11 supergravity. The operators associated with the other 4-moduli (also G_{21} -invariant) break supersymmetry.

Let us name

$$\mathbf{Y}^{\mathcal{N}=1}(\mathbf{x}|\psi) = \sum_{\alpha=1}^{14} \sum_{q=1}^{35} \mathbf{e}_{q\alpha}^{\mathcal{N}=1}(\psi) \mathbf{d}_q \times \hat{f}_{\alpha}(\mathbf{x}) \quad (9.21)$$

the Englert solution leading to $\mathcal{N} = 1$ supersymmetry and dependent on the 4-parameters ψ where, for

simplicity, in the last stage we have renamed, $\psi_5 \rightarrow \psi_4$. The unique shortest way of displaying the result is that of displaying the 35×14 matrix $\mathfrak{E}_{q\alpha}^{N=1}(\psi)$, dependent on the four moduli ψ_i , ($i = 1, \dots, 4$). The output is still ominously big, yet due to the relevance of the final result for further uses, we believe that it is worth showing it: furthermore this is the only way to give concreteness to the results we have obtained. The matrix $\mathfrak{E}_{q\alpha}^{N=1}(\psi)$ is displayed in appendix B.

With some ingenuity we have also found that there are two particular nice points in the 4-dimensional moduli space of this solution where the rank of all the 14 matrices $\mathcal{Q}_\alpha(\psi)$, coefficients of the basis functions, reduces from 7 to 4, although these matrices do not admit a common null-vector, leaving the rank of the full operator $\mathcal{B}(\mathbf{x}|\Psi)$ equal to 7. These two special points in moduli space might have some so far unknown deep significance and for this reason we feel it important to mention them:

$$\vec{\psi}_1 \equiv \{\psi_1 = 1, \psi_2 = -1, \psi_3 = -2, \psi_4 = 1\} \quad (9.22)$$

$$\vec{\psi}_2 \equiv \left\{ \psi_1 = 1, \psi_2 = -1, \psi_3 = \frac{1}{2}, \psi_4 = 1 \right\} \quad (9.23)$$

In the case of the point $\vec{\psi}_1$ in moduli space, the 35×14 matrix encoding the Englert solution takes

the following relatively simple look:

$$\mathfrak{C}_{q\alpha}^{\mathcal{N}=1}(\vec{\psi}_1) = \begin{pmatrix} 0 & 0 & -3 & 0 & 9 & -15 & 9 & -\frac{24}{\sqrt{7}} & 0 & \frac{39}{\sqrt{7}} & 0 & \frac{27}{\sqrt{7}} & -3\sqrt{7} & -3\sqrt{7} \\ -9 & -3 & 3 & -6 & 0 & 6 & 9 & 3\sqrt{7} & \frac{15}{\sqrt{7}} & -\frac{15}{\sqrt{7}} & \frac{6}{\sqrt{7}} & 0 & -\frac{6}{\sqrt{7}} & -3\sqrt{7} \\ 9 & 0 & -6 & -6 & 12 & -9 & 0 & \frac{3}{\sqrt{7}} & 0 & \frac{6}{\sqrt{7}} & \frac{6}{\sqrt{7}} & -\frac{60}{\sqrt{7}} & \frac{45}{\sqrt{7}} & 0 \\ 3 & -3 & 6 & -6 & 15 & 3 & -18 & -\frac{15}{\sqrt{7}} & \frac{15}{\sqrt{7}} & -\frac{6}{\sqrt{7}} & \frac{6}{\sqrt{7}} & -\frac{3}{\sqrt{7}} & -\frac{39}{\sqrt{7}} & 6\sqrt{7} \\ 12 & -3 & -12 & 0 & -15 & 18 & 0 & -\frac{12}{\sqrt{7}} & \frac{15}{\sqrt{7}} & \frac{12}{\sqrt{7}} & 0 & \frac{3}{\sqrt{7}} & -\frac{18}{\sqrt{7}} & 0 \\ 0 & -3 & -6 & 15 & 0 & -15 & 9 & -\frac{24}{\sqrt{7}} & \frac{39}{\sqrt{7}} & -6\sqrt{7} & 3\sqrt{7} & 0 & \frac{3}{\sqrt{7}} & \frac{3}{\sqrt{7}} \\ 0 & -9 & 3 & 15 & -9 & -12 & 12 & 0 & -\frac{27}{\sqrt{7}} & \frac{33}{\sqrt{7}} & 3\sqrt{7} & -\frac{27}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & \frac{12}{\sqrt{7}} \\ 6 & 6 & -3 & 15 & 0 & -6 & -18 & \frac{18}{\sqrt{7}} & -\frac{6}{\sqrt{7}} & -\frac{33}{\sqrt{7}} & 3\sqrt{7} & 0 & \frac{30}{\sqrt{7}} & -\frac{30}{\sqrt{7}} \\ 6 & -3 & 12 & 0 & 9 & -12 & -12 & \frac{18}{\sqrt{7}} & -\frac{33}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & 0 & \frac{27}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & \frac{12}{\sqrt{7}} \\ -9 & 12 & -18 & 18 & 0 & 3 & -6 & -\frac{3}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & \frac{18}{\sqrt{7}} & -\frac{18}{\sqrt{7}} & 0 & -\frac{15}{\sqrt{7}} & \frac{30}{\sqrt{7}} \\ 12 & -6 & -9 & 9 & 0 & -6 & 0 & -\frac{12}{\sqrt{7}} & \frac{6}{\sqrt{7}} & \frac{45}{\sqrt{7}} & -\frac{45}{\sqrt{7}} & 0 & \frac{30}{\sqrt{7}} & -\frac{24}{\sqrt{7}} \\ 3 & 6 & -15 & -3 & 0 & 3 & 6 & -\frac{15}{\sqrt{7}} & -\frac{6}{\sqrt{7}} & \frac{3}{\sqrt{7}} & \frac{39}{\sqrt{7}} & 0 & -\frac{15}{\sqrt{7}} & -\frac{6}{\sqrt{7}} \\ -3 & 6 & 0 & -9 & 15 & -6 & -3 & -\frac{9}{\sqrt{7}} & -\frac{6}{\sqrt{7}} & 0 & -\frac{27}{\sqrt{7}} & -\frac{3}{\sqrt{7}} & \frac{30}{\sqrt{7}} & \frac{15}{\sqrt{7}} \\ -3 & 6 & 12 & -3 & -3 & 0 & -9 & -\frac{9}{\sqrt{7}} & -\frac{6}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & \frac{39}{\sqrt{7}} & -\frac{57}{\sqrt{7}} & 0 & \frac{45}{\sqrt{7}} \\ 3 & -6 & -12 & -3 & 15 & 6 & -3 & \frac{9}{\sqrt{7}} & \frac{6}{\sqrt{7}} & \frac{12}{\sqrt{7}} & \frac{39}{\sqrt{7}} & -\frac{3}{\sqrt{7}} & -\frac{30}{\sqrt{7}} & -\frac{33}{\sqrt{7}} \\ 9 & 6 & 0 & -15 & -9 & 0 & 9 & -3\sqrt{7} & 6\sqrt{7} & 0 & \frac{3}{\sqrt{7}} & -\frac{27}{\sqrt{7}} & 0 & \frac{3}{\sqrt{7}} \\ -9 & 0 & -6 & 12 & 0 & 15 & -12 & -\frac{51}{\sqrt{7}} & 0 & \frac{6}{\sqrt{7}} & \frac{12}{\sqrt{7}} & 0 & 3\sqrt{7} & \frac{12}{\sqrt{7}} \\ 6 & 6 & 3 & 3 & 0 & 0 & -18 & -\frac{30}{\sqrt{7}} & 6\sqrt{7} & \frac{33}{\sqrt{7}} & -\frac{15}{\sqrt{7}} & 0 & 0 & -\frac{30}{\sqrt{7}} \\ -3 & 6 & -9 & 9 & 0 & -15 & 12 & -\frac{33}{\sqrt{7}} & 6\sqrt{7} & -\frac{27}{\sqrt{7}} & \frac{27}{\sqrt{7}} & 0 & -3\sqrt{7} & \frac{12}{\sqrt{7}} \\ -18 & 15 & -3 & 3 & 3 & -6 & 6 & 6\sqrt{7} & -\frac{3}{\sqrt{7}} & \frac{15}{\sqrt{7}} & -\frac{15}{\sqrt{7}} & -\frac{39}{\sqrt{7}} & \frac{6}{\sqrt{7}} & -\frac{6}{\sqrt{7}} \\ 0 & -18 & -3 & -3 & 24 & 0 & 0 & 0 & \frac{18}{\sqrt{7}} & \frac{15}{\sqrt{7}} & \frac{15}{\sqrt{7}} & -\frac{24}{\sqrt{7}} & 0 & -\frac{24}{\sqrt{7}} \\ 0 & -9 & 0 & 6 & 3 & 6 & -6 & 0 & \frac{45}{\sqrt{7}} & \frac{3}{\sqrt{7}} & 0 & -\frac{30}{\sqrt{7}} & -\frac{39}{\sqrt{7}} & -\frac{6}{\sqrt{7}} \\ 12 & -15 & -6 & 3 & 0 & 3 & 3 & \frac{12}{\sqrt{7}} & \frac{3}{\sqrt{7}} & \frac{6}{\sqrt{7}} & -\frac{15}{\sqrt{7}} & 0 & -\frac{39}{\sqrt{7}} & \frac{33}{\sqrt{7}} \\ -6 & -15 & 6 & -3 & 0 & 9 & 9 & \frac{6}{\sqrt{7}} & \frac{3}{\sqrt{7}} & -\frac{6}{\sqrt{7}} & \frac{15}{\sqrt{7}} & 0 & \frac{27}{\sqrt{7}} & -\frac{45}{\sqrt{7}} \\ -18 & 9 & 6 & -3 & 0 & 3 & 3 & \frac{18}{\sqrt{7}} & \frac{27}{\sqrt{7}} & -\frac{6}{\sqrt{7}} & \frac{15}{\sqrt{7}} & 0 & -\frac{39}{\sqrt{7}} & -\frac{15}{\sqrt{7}} \\ -9 & 12 & 6 & -12 & 0 & -3 & 6 & -\frac{51}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & 6\sqrt{7} & -\frac{12}{\sqrt{7}} & 0 & \frac{15}{\sqrt{7}} & \frac{18}{\sqrt{7}} \\ 3 & -3 & 6 & 12 & -9 & -9 & 0 & \frac{33}{\sqrt{7}} & -\frac{33}{\sqrt{7}} & 6\sqrt{7} & \frac{12}{\sqrt{7}} & -\frac{27}{\sqrt{7}} & -\frac{27}{\sqrt{7}} & 0 \\ -6 & 3 & 0 & 6 & -9 & 12 & -6 & \frac{30}{\sqrt{7}} & \frac{33}{\sqrt{7}} & 0 & -\frac{30}{\sqrt{7}} & \frac{12}{\sqrt{7}} & -\frac{18}{\sqrt{7}} & -\frac{18}{\sqrt{7}} \\ 3 & 15 & 3 & -12 & 0 & -6 & -3 & \frac{9}{\sqrt{7}} & 3\sqrt{7} & \frac{33}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & 0 & -6\sqrt{7} & -\frac{9}{\sqrt{7}} \\ 3 & -3 & -15 & -12 & 0 & 12 & 15 & \frac{9}{\sqrt{7}} & -\frac{33}{\sqrt{7}} & -3\sqrt{7} & -\frac{12}{\sqrt{7}} & 0 & \frac{12}{\sqrt{7}} & \frac{45}{\sqrt{7}} \\ -15 & -3 & 3 & 6 & 0 & 12 & -3 & -\frac{45}{\sqrt{7}} & -\frac{33}{\sqrt{7}} & \frac{33}{\sqrt{7}} & 6\sqrt{7} & 0 & \frac{12}{\sqrt{7}} & -\frac{9}{\sqrt{7}} \\ -6 & -12 & 9 & 0 & 3 & 3 & 3 & \frac{6}{\sqrt{7}} & \frac{12}{\sqrt{7}} & \frac{27}{\sqrt{7}} & 0 & -\frac{39}{\sqrt{7}} & -\frac{15}{\sqrt{7}} & \frac{9}{\sqrt{7}} \\ 12 & 12 & -15 & -6 & 3 & -3 & -3 & \frac{12}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & \frac{3}{\sqrt{7}} & \frac{30}{\sqrt{7}} & -\frac{39}{\sqrt{7}} & \frac{15}{\sqrt{7}} & -\frac{9}{\sqrt{7}} \\ -18 & 0 & -15 & 6 & 21 & 3 & 3 & \frac{18}{\sqrt{7}} & 0 & \frac{3}{\sqrt{7}} & -\frac{30}{\sqrt{7}} & \frac{15}{\sqrt{7}} & -\frac{15}{\sqrt{7}} & \frac{9}{\sqrt{7}} \\ 6 & 12 & -12 & -6 & 0 & 6 & -6 & \frac{18}{\sqrt{7}} & -\frac{12}{\sqrt{7}} & \frac{12}{\sqrt{7}} & \frac{30}{\sqrt{7}} & 0 & -\frac{30}{\sqrt{7}} & -\frac{18}{\sqrt{7}} \end{pmatrix} \quad (9.24)$$

9.4 The inhomogeneous harmonic function

The final step in our construction regards the explicit form of the inhomogeneous harmonic function. To this effect we have to construct the source term of eq.(2.32) starting from the explicit solution of Englert equation we have derived.

$$J(U, \mathbf{x}) \equiv \frac{3}{2} e^{-2U\mu} \mu^2 \sum_{ijk} Y_{ijk}^2(\mathbf{x}) \quad (9.25)$$

Focusing for simplicity on the case of the solution $\mathbf{Y}^{\mathcal{N}=1}(\mathbf{x}|\vec{\psi}_1)$ we find that the source term has the following form:

$$J = -1152 e^{4\sqrt{14}\pi U} \pi^2 (-441 + \mathfrak{W}(\mathbf{x})) \quad (9.26)$$

Where the function $\mathfrak{W}(x)$ is the following linear combination of 42 independent trigonometric functions:

$$\begin{aligned}
\mathfrak{W}(x) = & 21\text{Cos}\left[2\sqrt{2}\pi x_1\right] + 21\text{Cos}\left[2\sqrt{2}\pi x_2\right] + 21\text{Cos}\left[2\sqrt{2}\pi x_4\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 - x_4 - x_5 - x_6)\right] \\
& + 21\text{Cos}\left[\sqrt{2}\pi(x_1 + x_4 - x_5 - x_6)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_2 + x_3 + x_5 - x_6)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_2 - x_3 - x_5 + x_6)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 - x_4 + x_5 + x_6)\right] \\
& + 21\text{Cos}\left[\sqrt{2}\pi(x_1 + x_4 + x_5 + x_6)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 - x_2 + x_5 - x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 + x_2 + x_5 - x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_2 - x_4 - x_6 - x_7)\right] \\
& + 21\text{Cos}\left[\sqrt{2}\pi(x_2 + x_4 - x_6 - x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 - x_3 + x_6 - x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 - x_2 - x_5 + x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 + x_2 - x_5 + x_7)\right] \\
& + 21\text{Cos}\left[\sqrt{2}\pi(x_3 - x_4 + x_5 + x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_3 + x_4 + x_5 + x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_1 + x_3 - x_6 + x_7)\right] + 21\text{Cos}\left[\sqrt{2}\pi(x_2 - x_4 + x_6 + x_7)\right] \\
& + 21\text{Cos}\left[\sqrt{2}\pi(x_2 + x_4 + x_6 + x_7)\right] + \sqrt{7}\text{Sin}\left[2\sqrt{2}\pi x_1\right] - \sqrt{7}\text{Sin}\left[2\sqrt{2}\pi x_2\right] - \sqrt{7}\text{Sin}\left[2\sqrt{2}\pi x_4\right] - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 - x_4 - x_5 - x_6)\right] \\
& + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 + x_4 - x_5 - x_6)\right] - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_2 + x_3 + x_5 - x_6)\right] - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_2 - x_3 - x_5 + x_6)\right] \\
& + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 - x_4 + x_5 + x_6)\right] - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 + x_4 + x_5 + x_6)\right] - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 - x_2 + x_5 - x_7)\right] \\
& + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(-x_1 + x_2 + x_5 - x_7)\right] - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 + x_2 + x_5 - x_7)\right] + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_2 - x_4 - x_6 - x_7)\right] \\
& + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_2 + x_4 - x_6 - x_7)\right] + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 - x_3 + x_6 - x_7)\right] \\
& + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 + x_2 - x_5 + x_7)\right] + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_3 - x_4 + x_5 + x_7)\right] + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_3 + x_4 + x_5 + x_7)\right] \\
& - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_1 + x_3 - x_6 + x_7)\right] - \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_2 - x_4 + x_6 + x_7)\right] + \sqrt{7}\text{Sin}\left[\sqrt{2}\pi(x_2 + x_4 + x_6 + x_7)\right]
\end{aligned} \tag{9.27}$$

The function $\mathfrak{W}(x)$ is an eigenstate of the Laplacian on the seven torus:

$$\square_{T^7} \mathfrak{W}(x) = -8\pi^2 \mathfrak{W}(x) \tag{9.28}$$

Hence the equation

$$\square_{\mathbb{R}_+ \otimes T^7} H(U, x) + J(U, x) = 0 \tag{9.29}$$

admits the following solution:

$$H(U, x) = \alpha - 2268 e^{4\sqrt{14}\pi U} + \frac{16}{3} e^{4\sqrt{14}\pi U} \mathfrak{W}(x) \tag{9.30}$$

where α is an arbitrary constant that can be fixed by boundary conditions.

In this way we have completed the derivation of an M2-brane solution of $d = 11$ supergravity that preserves $\mathcal{N} = 1$ supersymmetry in $d = 3$ counting and has an exact G_{21} discrete symmetry. It should be noted that the metric has a quite non trivial dependence on all the coordinates of the space $\mathbb{R}_+ \times T^7$, transverse to the brane volume. This together with the structure of the Englert fields implies that the fields of the $d = 3$ brane model are all strongly, non linearly interacting.

10 Conclusions

In this paper we have shown that, contrary to what happens in compactifications of the type:

$$\mathcal{M}_{11} = \text{AdS}_4 \times \mathcal{M}_7 \tag{10.1}$$

there exist M2-brane supersymmetric solutions of $d = 11$ supergravity with internal fluxes governed by the Englert equation. In the case of eq.(10.1) Englert solutions exist but they always break supersymmetry. We have identified Englert equation as the proper generalization to 7-manifolds of Beltrami equation defined on 3-manifolds. We have shown a general procedure to construct M2-brane solutions with Englert fluxes and we have defined a simple and algorithmic criterion to determine the

numebr of supersymmetries preserved by such backgrounds.

Building on our experience with the torus T^3 and the use of its crystallographic point group for the construction of solutions of Beltrami equation, we have spotted the simple group $L_{168} \equiv \text{PSL}(2, \mathbb{Z}_7)$ as a very much challenging crystallographic point-group in 7-dimensions, the corresponding lattice being the A7-root lattice. Relying on this we have defined an algorithm to construct solutions of Englert equation associated with orbits of L_{168} and of its subgroups in the weight-lattice of A7.

In this framework we have constructed a very non trivial M2-brane solution with $\mathcal{N} = 1$ -supersymmetry and a large non-abelian discrete symmetry, namely $G_{21} \equiv \mathbb{Z}_3 \ltimes \mathbb{Z}_7$.

The next obvious step is the analysis of the $d = 3$ theories on the world volume that are dual to $d = 11$ supergravity localized on the considered backgrounds. We plan to address this problem in future publications.

We note in passing that, as a by-product of our main investigation, we have classified all the orbits of $\text{PSL}(2, \mathbb{Z}_7)$ in the A7-weight lattice. This mathematical result might prove useful in different contests both mathematical and physical.

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A The explicit form of the L_{168} -invariant Englert solution

In this appendix we display the explicit form of the 2-moduli dependent coefficients $\mathfrak{C}_{q\alpha}^{[168]}(\hat{\mathbf{F}})$ that define the L_{168} -invariant solution of Englert equation discussed in section 8. Because of the large form of the output the entries of the matrix are organized in 4 tables containing the columns from 1 to 4, from 5 to 8, from 8 to 12 and finally from 13 to 16.

0	1	2	3	4
1	$6F_1$	$6F_1$	$-6F_1$	$-6F_1$
2	$6F_2$	$6F_2$	$6(2F_1 + F_2)$	$-6(4F_1 + F_2)$
3	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$
4	$6(F_1 + F_2)$	$6(3F_1 + F_2)$	$6(3F_1 + F_2)$	$-6(F_1 + 3F_2)$
5	$12F_1$	$-12F_1$	0	$36F_1 + 24F_2$
6	$-6(F_1 + F_2)$	$6(F_1 - F_2)$	$6(F_1 + F_2)$	$6(3F_1 + F_2)$
7	$6(5F_1 + 2F_2)$	$6(F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
8	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(5F_1 + 2F_2)$	$6(F_1 + 2F_2)$
9	$6(F_1 + 2F_2)$	$6(5F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
10	$-18F_1$	$18F_1$	$6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
11	$6(5F_1 + 2F_2)$	$6(F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(5F_1 + 2F_2)$
12	$-18(2F_1 + F_2)$	$-18(2F_1 + F_2)$	$6(F_2 - 2F_1)$	$6(F_2 - 2F_1)$
13	$-6(4F_1 + F_2)$	$-6(2F_1 + F_2)$	$6(F_2 - 2F_1)$	$-6(8F_1 + 3F_2)$
14	$18F_1$	$18F_1$	$42F_1$	$42F_1$
15	$6F_2 - 6F_1$	$6(F_1 + F_2)$	$6(F_1 + 3F_2)$	$-6(5F_1 + F_2)$
16	$6(3F_1 + 2F_2)$	$6F_1$	$6F_1$	$-6F_1$
17	$-6(8F_1 + 3F_2)$	$6F_2$	$6F_2$	$6(2F_1 + F_2)$
18	$6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
19	$-6(F_1 + 3F_2)$	$6(3F_1 + F_2)$	$6(F_1 + F_2)$	$6(3F_1 + F_2)$
20	$6(5F_1 + F_2)$	$-6(F_1 + F_2)$	$6(F_1 - F_2)$	$6(F_1 + F_2)$
21	$-6(3F_1 + 2F_2)$	$6(5F_1 + 2F_2)$	$6(F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
22	$36F_1 + 24F_2$	$-12F_1$	$12F_1$	$12F_1$
23	$6(3F_1 + 2F_2)$	$-18F_1$	$18F_1$	$6(3F_1 + 2F_2)$
24	$-6(5F_1 + F_2)$	$-6(F_1 + F_2)$	$-6(5F_1 + F_2)$	$-6(5F_1 + F_2)$
25	$6(5F_1 + F_2)$	$-6(F_1 + F_2)$	$6(F_1 - F_2)$	$6(5F_1 + F_2)$
26	$-6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$	$6F_1$	$6F_1$
27	$6(F_2 - 2F_1)$	$-6(8F_1 + 3F_2)$	$6F_2$	$6F_2$
28	$12F_1 - 6F_2$	$12F_1 - 6F_2$	$-6F_2$	$-6(2F_1 + F_2)$
29	$-6(F_1 + 3F_2)$	$6(5F_1 + F_2)$	$-6(F_1 + F_2)$	$6(F_1 - F_2)$
30	$6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$
31	$-6(8F_1 + 3F_2)$	$6(F_2 - 2F_1)$	$-6(4F_1 + F_2)$	$-6(2F_1 + F_2)$
32	$6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$	$6F_1$
33	0	0	$-12(3F_1 + 2F_2)$	0
34	$12F_1 - 6F_2$	$12F_1 - 6F_2$	$6(8F_1 + 3F_2)$	$6(2F_1 + F_2)$
35	$6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$

(A.1)

0	5	6	7	8
1	$6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$
2	$-6(2F_1 + F_2)$	$6(F_2 - 2F_1)$	$6(F_2 - 2F_1)$	$-6(8F_1 + 3F_2)$
3	$-18F_1$	$18F_1$	$6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$
4	$6(5F_1 + F_2)$	$-6(F_1 + F_2)$	$6(5F_1 + F_2)$	$6(F_1 - F_2)$
5	0	0	$-12(2F_1 + F_2)$	$-12(F_1 + F_2)$
6	$6(3F_1 + F_2)$	$-6(F_1 + 3F_2)$	$6(5F_1 + F_2)$	$6(5F_1 + F_2)$
7	$6(5F_1 + 2F_2)$	$6(F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
8	$-6(3F_1 + 2F_2)$	$6(5F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(F_1 + 2F_2)$
9	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(F_1 + 2F_2)$	$6(5F_1 + 2F_2)$
10	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$
11	$6(F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
12	$6(2F_1 + F_2)$	$6(F_2 - 2F_1)$	$6(2F_1 + F_2)$	$6(F_2 - 2F_1)$
13	$6F_2$	$6F_2$	$6(F_2 - 2F_1)$	$6(2F_1 + F_2)$
14	$18F_1$	$18F_1$	$18F_1$	$18F_1$
15	$-6(5F_1 + F_2)$	$-6(F_1 + F_2)$	$-6(3F_1 + F_2)$	$-6(3F_1 + F_2)$
16	$-6F_1$	$6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
17	$-6(4F_1 + F_2)$	$-6(2F_1 + F_2)$	$6(F_2 - 2F_1)$	$6(F_2 - 2F_1)$
18	$6(3F_1 + 2F_2)$	$-18F_1$	$6(3F_1 + 2F_2)$	$18F_1$
19	$6(5F_1 + F_2)$	$6(5F_1 + F_2)$	$6(F_1 - F_2)$	$-6(F_1 + F_2)$
20	$6(3F_1 + F_2)$	$6(3F_1 + F_2)$	$6(5F_1 + F_2)$	$-6(F_1 + 3F_2)$
21	$-6(3F_1 + 2F_2)$	$6(5F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(F_1 + 2F_2)$
22	$-12F_1$	$-12(3F_1 + 2F_2)$	$-12F_1$	$12F_1$
23	$-6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$	$-6(3F_1 + 2F_2)$
24	$18(F_1 + F_2)$	$18(F_1 + F_2)$	$-6(F_1 + F_2)$	$-6(5F_1 + F_2)$
25	$-6(F_1 + 3F_2)$	$6(3F_1 + F_2)$	$6(F_1 + F_2)$	$6(3F_1 + F_2)$
26	$-6F_1$	$-6F_1$	$-6(3F_1 + 2F_2)$	$6(3F_1 + 2F_2)$
27	$6(2F_1 + F_2)$	$-6(4F_1 + F_2)$	$6(F_2 - 2F_1)$	$-6(2F_1 + F_2)$
28	$-6F_2$	$6(8F_1 + 3F_2)$	$6(4F_1 + F_2)$	$6(2F_1 + F_2)$
29	$6(F_1 + F_2)$	$6(3F_1 + F_2)$	$6(5F_1 + F_2)$	$6(3F_1 + F_2)$
30	$-6(F_1 + 2F_2)$	$-6(5F_1 + 2F_2)$	$-6(5F_1 + 2F_2)$	$-6(F_1 + 2F_2)$
31	$6(F_2 - 2F_1)$	$6F_2$	$6(2F_1 + F_2)$	$6F_2$
32	$6F_1$	$-6F_1$	$-6(3F_1 + 2F_2)$	$-6F_1$
33	$-12F_1$	$12F_1$	$12(F_1 + F_2)$	$12(2F_1 + F_2)$
34	$6(4F_1 + F_2)$	$-6F_2$	$-6F_2$	$-6(2F_1 + F_2)$
35	$6F_1$	$-6F_1$	$-6F_1$	$6F_1$

(A.2)

0	9	10	11	12
1	$-\frac{6(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{6(3F_1+2F_2)}{\sqrt{7}}$	$\frac{6(3F_1+2F_2)}{\sqrt{7}}$	$\frac{6(3F_1+2F_2)}{\sqrt{7}}$
2	$\frac{6(8F_1+3F_2)}{\sqrt{7}}$	$\frac{6(8F_1+3F_2)}{\sqrt{7}}$	$\frac{6(2F_1-F_2)}{\sqrt{7}}$	$\frac{6(4F_1+5F_2)}{\sqrt{7}}$
3	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$	$6\sqrt{7}F_1$
4	$\frac{6(5F_1+F_2)}{\sqrt{7}}$	$-\frac{6(F_1+3F_2)}{\sqrt{7}}$	$-\frac{6(F_1+3F_2)}{\sqrt{7}}$	$-6\sqrt{7}(3F_1+F_2)$
5	$-\frac{12(3F_1+2F_2)}{\sqrt{7}}$	$\frac{12(3F_1+2F_2)}{\sqrt{7}}$	0	$12\sqrt{7}F_1$
6	$-\frac{6(5F_1+F_2)}{\sqrt{7}}$	$-\frac{6(11F_1+5F_2)}{\sqrt{7}}$	$\frac{6(5F_1+F_2)}{\sqrt{7}}$	$-\frac{6(F_1+3F_2)}{\sqrt{7}}$
7	$\frac{6(F_1-4F_2)}{\sqrt{7}}$	$\frac{6(13F_1+4F_2)}{\sqrt{7}}$	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$
8	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$	$\frac{6(F_1-4F_2)}{\sqrt{7}}$	$\frac{6(13F_1+4F_2)}{\sqrt{7}}$
9	$\frac{6(13F_1+4F_2)}{\sqrt{7}}$	$\frac{6(F_1-4F_2)}{\sqrt{7}}$	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$
10	$\frac{18(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{18(3F_1+2F_2)}{\sqrt{7}}$	$6\sqrt{7}F_1$	$-6\sqrt{7}F_1$
11	$\frac{6(F_1-4F_2)}{\sqrt{7}}$	$\frac{6(13F_1+4F_2)}{\sqrt{7}}$	$-6\sqrt{7}F_1$	$\frac{6(F_1-4F_2)}{\sqrt{7}}$
12	$\frac{18(F_2-2F_1)}{\sqrt{7}}$	$\frac{18(F_2-2F_1)}{\sqrt{7}}$	$6\sqrt{7}(2F_1+F_2)$	$6\sqrt{7}(2F_1+F_2)$
13	$\frac{6(4F_1+5F_2)}{\sqrt{7}}$	$\frac{6(F_2-2F_1)}{\sqrt{7}}$	$6\sqrt{7}(2F_1+F_2)$	$6\sqrt{7}F_2$
14	$-\frac{18(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{18(3F_1+2F_2)}{\sqrt{7}}$	$-6\sqrt{7}(3F_1+2F_2)$	$-6\sqrt{7}(3F_1+2F_2)$
15	$\frac{6(11F_1+5F_2)}{\sqrt{7}}$	$\frac{6(5F_1+F_2)}{\sqrt{7}}$	$6\sqrt{7}(3F_1+F_2)$	$6\sqrt{7}(F_1+F_2)$
16	$6\sqrt{7}F_1$	$-\frac{6(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{6(3F_1+2F_2)}{\sqrt{7}}$	$\frac{6(3F_1+2F_2)}{\sqrt{7}}$
17	$6\sqrt{7}F_2$	$\frac{6(8F_1+3F_2)}{\sqrt{7}}$	$\frac{6(8F_1+3F_2)}{\sqrt{7}}$	$\frac{6(2F_1-F_2)}{\sqrt{7}}$
18	$6\sqrt{7}F_1$	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$
19	$-6\sqrt{7}(3F_1+F_2)$	$-\frac{6(F_1+3F_2)}{\sqrt{7}}$	$\frac{6(5F_1+F_2)}{\sqrt{7}}$	$-\frac{6(F_1+3F_2)}{\sqrt{7}}$
20	$-6\sqrt{7}(F_1+F_2)$	$-\frac{6(5F_1+F_2)}{\sqrt{7}}$	$-\frac{6(11F_1+5F_2)}{\sqrt{7}}$	$\frac{6(5F_1+F_2)}{\sqrt{7}}$
21	$-6\sqrt{7}F_1$	$\frac{6(F_1-4F_2)}{\sqrt{7}}$	$\frac{6(13F_1+4F_2)}{\sqrt{7}}$	$-6\sqrt{7}F_1$
22	$12\sqrt{7}F_1$	$\frac{12(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{12(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{12(3F_1+2F_2)}{\sqrt{7}}$
23	$6\sqrt{7}F_1$	$\frac{18(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{18(3F_1+2F_2)}{\sqrt{7}}$	$6\sqrt{7}F_1$
24	$6\sqrt{7}(F_1+F_2)$	$-\frac{6(5F_1+F_2)}{\sqrt{7}}$	$6\sqrt{7}(F_1+F_2)$	$6\sqrt{7}(F_1+F_2)$
25	$-6\sqrt{7}(F_1+F_2)$	$-\frac{6(5F_1+F_2)}{\sqrt{7}}$	$-\frac{6(11F_1+5F_2)}{\sqrt{7}}$	$-6\sqrt{7}(F_1+F_2)$
26	$-6\sqrt{7}F_1$	$6\sqrt{7}F_1$	$-\frac{6(3F_1+2F_2)}{\sqrt{7}}$	$-\frac{6(3F_1+2F_2)}{\sqrt{7}}$
27	$6\sqrt{7}(2F_1+F_2)$	$6\sqrt{7}F_2$	$\frac{6(8F_1+3F_2)}{\sqrt{7}}$	$\frac{6(8F_1+3F_2)}{\sqrt{7}}$
28	$-6\sqrt{7}(2F_1+F_2)$	$-6\sqrt{7}(2F_1+F_2)$	$-\frac{6(8F_1+3F_2)}{\sqrt{7}}$	$\frac{6(F_2-2F_1)}{\sqrt{7}}$
29	$-6\sqrt{7}(3F_1+F_2)$	$-6\sqrt{7}(F_1+F_2)$	$-\frac{6(5F_1+F_2)}{\sqrt{7}}$	$-\frac{6(11F_1+5F_2)}{\sqrt{7}}$
30	$6\sqrt{7}F_1$	$6\sqrt{7}F_1$	$6\sqrt{7}F_1$	$6\sqrt{7}F_1$
31	$6\sqrt{7}F_2$	$6\sqrt{7}(2F_1+F_2)$	$\frac{6(4F_1+5F_2)}{\sqrt{7}}$	$\frac{6(F_2-2F_1)}{\sqrt{7}}$
32	$6\sqrt{7}F_1$	$-6\sqrt{7}F_1$	$6\sqrt{7}F_1$	$-\frac{6(3F_1+2F_2)}{\sqrt{7}}$
33	0	0	$-12\sqrt{7}F_1$	0
34	$-6\sqrt{7}(2F_1+F_2)$	$-6\sqrt{7}(2F_1+F_2)$	$-6\sqrt{7}F_2$	$\frac{6(2F_1-F_2)}{\sqrt{7}}$
35	$6\sqrt{7}F_1$	$6\sqrt{7}F_1$	$-6\sqrt{7}F_1$	$-6\sqrt{7}F_1$

(A.3)

$$\begin{pmatrix}
0 & 13 & 14 & 15 & 16 \\
1 & 6\sqrt{7}F_1 & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 & 6\sqrt{7}F_1 \\
2 & \frac{6(F_2-2F_1)}{\sqrt{7}} & 6\sqrt{7}(2F_1+F_2) & 6\sqrt{7}(2F_1+F_2) & 6\sqrt{7}F_2 \\
3 & \frac{18(3F_1+2F_2)}{\sqrt{7}} & -\frac{18(3F_1+2F_2)}{\sqrt{7}} & 6\sqrt{7}F_1 & 6\sqrt{7}F_1 \\
4 & -6\sqrt{7}(F_1+F_2) & -\frac{6(5F_1+F_2)}{\sqrt{7}} & -6\sqrt{7}(F_1+F_2) & -\frac{6(11F_1+5F_2)}{\sqrt{7}} \\
5 & 0 & 0 & \frac{12(F_2-2F_1)}{\sqrt{7}} & -\frac{12(5F_1+F_2)}{\sqrt{7}} \\
6 & -\frac{6(F_1+3F_2)}{\sqrt{7}} & -6\sqrt{7}(3F_1+F_2) & -6\sqrt{7}(F_1+F_2) & -6\sqrt{7}(F_1+F_2) \\
7 & \frac{6(F_1-4F_2)}{\sqrt{7}} & \frac{6(13F_1+4F_2)}{\sqrt{7}} & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 \\
8 & -6\sqrt{7}F_1 & \frac{6(F_1-4F_2)}{\sqrt{7}} & -6\sqrt{7}F_1 & \frac{6(13F_1+4F_2)}{\sqrt{7}} \\
9 & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 & \frac{6(13F_1+4F_2)}{\sqrt{7}} & \frac{6(F_1-4F_2)}{\sqrt{7}} \\
10 & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 & 6\sqrt{7}F_1 & 6\sqrt{7}F_1 \\
11 & \frac{6(13F_1+4F_2)}{\sqrt{7}} & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 \\
12 & \frac{6(2F_1-F_2)}{\sqrt{7}} & 6\sqrt{7}(2F_1+F_2) & \frac{6(2F_1-F_2)}{\sqrt{7}} & 6\sqrt{7}(2F_1+F_2) \\
13 & \frac{6(8F_1+3F_2)}{\sqrt{7}} & \frac{6(8F_1+3F_2)}{\sqrt{7}} & 6\sqrt{7}(2F_1+F_2) & \frac{6(2F_1-F_2)}{\sqrt{7}} \\
14 & -\frac{18(3F_1+2F_2)}{\sqrt{7}} & -\frac{18(3F_1+2F_2)}{\sqrt{7}} & -\frac{18(3F_1+2F_2)}{\sqrt{7}} & -\frac{18(3F_1+2F_2)}{\sqrt{7}} \\
15 & 6\sqrt{7}(F_1+F_2) & -\frac{6(5F_1+F_2)}{\sqrt{7}} & \frac{6(F_1+3F_2)}{\sqrt{7}} & \frac{6(F_1+3F_2)}{\sqrt{7}} \\
16 & \frac{6(3F_1+2F_2)}{\sqrt{7}} & 6\sqrt{7}F_1 & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 \\
17 & \frac{6(4F_1+5F_2)}{\sqrt{7}} & \frac{6(F_2-2F_1)}{\sqrt{7}} & 6\sqrt{7}(2F_1+F_2) & 6\sqrt{7}(2F_1+F_2) \\
18 & 6\sqrt{7}F_1 & \frac{18(3F_1+2F_2)}{\sqrt{7}} & 6\sqrt{7}F_1 & -\frac{18(3F_1+2F_2)}{\sqrt{7}} \\
19 & -6\sqrt{7}(F_1+F_2) & -6\sqrt{7}(F_1+F_2) & -\frac{6(11F_1+5F_2)}{\sqrt{7}} & -\frac{6(5F_1+F_2)}{\sqrt{7}} \\
20 & -\frac{6(F_1+3F_2)}{\sqrt{7}} & -\frac{6(F_1+3F_2)}{\sqrt{7}} & -6\sqrt{7}(F_1+F_2) & -6\sqrt{7}(3F_1+F_2) \\
21 & -6\sqrt{7}F_1 & \frac{6(F_1-4F_2)}{\sqrt{7}} & -6\sqrt{7}F_1 & \frac{6(13F_1+4F_2)}{\sqrt{7}} \\
22 & \frac{12(3F_1+2F_2)}{\sqrt{7}} & -12\sqrt{7}F_1 & \frac{12(3F_1+2F_2)}{\sqrt{7}} & -\frac{12(3F_1+2F_2)}{\sqrt{7}} \\
23 & -6\sqrt{7}F_1 & -6\sqrt{7}F_1 & 6\sqrt{7}F_1 & -6\sqrt{7}F_1 \\
24 & \frac{18(5F_1+F_2)}{\sqrt{7}} & \frac{18(5F_1+F_2)}{\sqrt{7}} & -\frac{6(5F_1+F_2)}{\sqrt{7}} & 6\sqrt{7}(F_1+F_2) \\
25 & -6\sqrt{7}(3F_1+F_2) & -\frac{6(F_1+3F_2)}{\sqrt{7}} & \frac{6(5F_1+F_2)}{\sqrt{7}} & -\frac{6(F_1+3F_2)}{\sqrt{7}} \\
26 & \frac{6(3F_1+2F_2)}{\sqrt{7}} & \frac{6(3F_1+2F_2)}{\sqrt{7}} & -6\sqrt{7}F_1 & 6\sqrt{7}F_1 \\
27 & \frac{6(2F_1-F_2)}{\sqrt{7}} & \frac{6(4F_1+5F_2)}{\sqrt{7}} & 6\sqrt{7}(2F_1+F_2) & \frac{6(F_2-2F_1)}{\sqrt{7}} \\
28 & -\frac{6(8F_1+3F_2)}{\sqrt{7}} & -6\sqrt{7}F_2 & -\frac{6(4F_1+5F_2)}{\sqrt{7}} & \frac{6(2F_1-F_2)}{\sqrt{7}} \\
29 & \frac{6(5F_1+F_2)}{\sqrt{7}} & -\frac{6(F_1+3F_2)}{\sqrt{7}} & -6\sqrt{7}(F_1+F_2) & -\frac{6(F_1+3F_2)}{\sqrt{7}} \\
30 & -\frac{6(13F_1+4F_2)}{\sqrt{7}} & -\frac{6(F_1-4F_2)}{\sqrt{7}} & -\frac{6(F_1-4F_2)}{\sqrt{7}} & -\frac{6(13F_1+4F_2)}{\sqrt{7}} \\
31 & 6\sqrt{7}(2F_1+F_2) & \frac{6(8F_1+3F_2)}{\sqrt{7}} & \frac{6(2F_1-F_2)}{\sqrt{7}} & \frac{6(8F_1+3F_2)}{\sqrt{7}} \\
32 & -\frac{6(3F_1+2F_2)}{\sqrt{7}} & \frac{6(3F_1+2F_2)}{\sqrt{7}} & -6\sqrt{7}F_1 & \frac{6(3F_1+2F_2)}{\sqrt{7}} \\
33 & \frac{12(3F_1+2F_2)}{\sqrt{7}} & -\frac{12(3F_1+2F_2)}{\sqrt{7}} & \frac{12(5F_1+F_2)}{\sqrt{7}} & \frac{12(2F_1-F_2)}{\sqrt{7}} \\
34 & -\frac{6(4F_1+5F_2)}{\sqrt{7}} & -\frac{6(8F_1+3F_2)}{\sqrt{7}} & -\frac{6(8F_1+3F_2)}{\sqrt{7}} & \frac{6(F_2-2F_1)}{\sqrt{7}} \\
35 & -\frac{6(3F_1+2F_2)}{\sqrt{7}} & \frac{6(3F_1+2F_2)}{\sqrt{7}} & \frac{6(3F_1+2F_2)}{\sqrt{7}} & -\frac{6(3F_1+2F_2)}{\sqrt{7}}
\end{pmatrix} \tag{A.4}$$

B The Explicit form of the 35×14 coefficient matrix $\mathfrak{C}_{q\alpha}(\psi)$ for the $\mathcal{N} = 1$ Englert 3-form

In this appendix we display the explicit form of the 4-moduli dependent coefficients that define our 4-parameter dependent solution of Englert equation leading to an $N = 1$ supergravity solution invariant under the discrete group G_{21} . Because of the large form of the output the entries of the matrix are organized in 4

tables containing the first columns from 1 to 4, from 5 to 8, from 8 to 12 and finally from 13 to 14.

0	1	2	3	4
1	$\frac{3(6\psi_1+\psi_2+2\psi_3-\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_2+\psi_4)$	$-\frac{3(9\psi_1+2\psi_2+2\psi_3-\psi_4)}{\sqrt{2}}$	$-\frac{3(\psi_1+5\psi_2+4\psi_4)}{\sqrt{2}}$
2	$-\frac{3(4\psi_1+3\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1+3\psi_2+2\psi_4)$	$\frac{3(5\psi_1+6\psi_2+3\psi_4)}{\sqrt{2}}$	$\frac{3(3\psi_1+\psi_2+2\psi_3-2\psi_4)}{\sqrt{2}}$
3	$-\frac{3(6\psi_1+3\psi_2+4\psi_3-\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1+\psi_2)$	$\frac{3(\psi_1+2\psi_2+2\psi_3+\psi_4)}{\sqrt{2}}$	$\frac{3(\psi_1-\psi_2+2\psi_3-2\psi_4)}{\sqrt{2}}$
4	$3\sqrt{2}(2\psi_1+3\psi_2+2\psi_4)$	$-3\sqrt{2}\psi_1$	$-3\sqrt{2}(\psi_1-\psi_2+\psi_3-2\psi_4)$	$-3\sqrt{2}(\psi_2-\psi_3+\psi_4)$
5	$-\frac{3(2\psi_1-3\psi_2+4\psi_3-5\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(2\psi_1+\psi_2)$	$\frac{3(3\psi_1+4\psi_3-3\psi_4)}{\sqrt{2}}$	$\frac{3(\psi_1+\psi_2)}{\sqrt{2}}$
6	$\frac{3(8\psi_1+5\psi_2+2\psi_3+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(3\psi_1-\psi_2+\psi_3-\psi_4)$	$\frac{3(15\psi_1+6\psi_2+6\psi_3-\psi_4)}{\sqrt{2}}$	$-\frac{3(13\psi_1+\psi_2+2\psi_3-6\psi_4)}{\sqrt{2}}$
7	$3\sqrt{2}(\psi_2+\psi_4)$	$3\sqrt{2}(5\psi_1-\psi_2+3\psi_3-3\psi_4)$	$-3\sqrt{2}(5\psi_1+2\psi_3-2\psi_4)$	$-3\sqrt{2}(4\psi_1-5\psi_2+4\psi_3-6\psi_4)$
8	$-\frac{3(6\psi_1+\psi_2+4\psi_3-\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(\psi_2-\psi_3+\psi_4)$	$\frac{3(9\psi_1+2\psi_2+4\psi_3-\psi_4)}{\sqrt{2}}$	$-\frac{3(13\psi_1+\psi_2+8\psi_3-6\psi_4)}{\sqrt{2}}$
9	$\frac{3(-6\psi_1+\psi_2-4\psi_3+3\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(5\psi_1+2\psi_3-2\psi_4)$	$-\frac{3(\psi_1-2\psi_2+4\psi_3-3\psi_4)}{\sqrt{2}}$	$\frac{3(3\psi_1+\psi_2-2\psi_4)}{\sqrt{2}}$
10	$\frac{3(4\psi_1+\psi_2+4\psi_3-\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1-2\psi_2+2\psi_3-3\psi_4)$	$\frac{3(\psi_1-2\psi_2+6\psi_3-3\psi_4)}{\sqrt{2}}$	$-\frac{3(5\psi_1-3\psi_2+6\psi_3-8\psi_4)}{\sqrt{2}}$
11	$\frac{3(5\psi_2-4\psi_3+5\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(\psi_1-\psi_2+\psi_3-2\psi_4)$	$-\frac{3(11\psi_1+12\psi_2+7\psi_4)}{\sqrt{2}}$	$\frac{3(7\psi_1-\psi_2-2\psi_4)}{\sqrt{2}}$
12	$3\sqrt{2}(2\psi_1+3\psi_2+2\psi_4)$	$3\sqrt{2}(\psi_2-\psi_3+\psi_4)$	$3\sqrt{2}(\psi_1-\psi_2+3\psi_3-\psi_4)$	$-3\sqrt{2}(3\psi_1-3\psi_2+\psi_3-3\psi_4)$
13	$\frac{6\psi_1-3(\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1+\psi_2+\psi_3)$	$\frac{9\psi_1+6\psi_2-3\psi_4}{\sqrt{2}}$	$\frac{3(9\psi_1-\psi_2+6\psi_3-4\psi_4)}{\sqrt{2}}$
14	$\frac{6\psi_1-3(\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1+\psi_2+\psi_3)$	$-\frac{3(5\psi_1+6\psi_2+4\psi_3+\psi_4)}{\sqrt{2}}$	$-\frac{3(7\psi_1+\psi_2+2\psi_3)}{\sqrt{2}}$
15	$\frac{3(-2\psi_1+\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(\psi_1+\psi_2+\psi_3)$	$\frac{3(\psi_1-6\psi_2+4\psi_3-7\psi_4)}{\sqrt{2}}$	$-\frac{3(9\psi_1+3\psi_2+2\psi_3)}{\sqrt{2}}$
16	$-\frac{3(5\psi_2+2\psi_3+3\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(5\psi_1+\psi_2+3\psi_3)$	$\frac{3(\psi_1-2\psi_2-3\psi_4)}{\sqrt{2}}$	$\frac{3(5\psi_1-\psi_2+6\psi_3-4\psi_4)}{\sqrt{2}}$
17	$\frac{3(18\psi_1+\psi_2+8\psi_3-7\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1+2\psi_2+\psi_4)$	$\frac{9\psi_1+6\psi_2+6\psi_3-3\psi_4}{\sqrt{2}}$	$-\frac{3(5\psi_1-3\psi_2+6\psi_3-4\psi_4)}{\sqrt{2}}$
18	$\frac{3(2\psi_1-\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(7\psi_1+\psi_2+3\psi_3-2\psi_4)$	$-\frac{3(7\psi_1-4\psi_2+4\psi_3-5\psi_4)}{\sqrt{2}}$	$\frac{3(5\psi_1+5\psi_2+2\psi_4)}{\sqrt{2}}$
19	$3\sqrt{2}(4\psi_1-\psi_2+2\psi_3-2\psi_4)$	$-3\sqrt{2}(8\psi_1+5\psi_2+3\psi_3+\psi_4)$	$3\sqrt{2}(5\psi_1-\psi_2+3\psi_3-3\psi_4)$	$-3\sqrt{2}(5\psi_1+\psi_2+3\psi_3-\psi_4)$
20	$-3\sqrt{2}(2\psi_1+\psi_2-2\psi_3+\psi_4)$	$-3\sqrt{2}(\psi_1-5\psi_2+3\psi_3-5\psi_4)$	$-3\sqrt{2}\psi_1$	$3\sqrt{2}(2\psi_1+3\psi_2+2\psi_4)$
21	$\frac{3(\psi_2+\psi_4)}{\sqrt{2}}$	$-9\sqrt{2}(\psi_2-\psi_3+\psi_4)$	$-\frac{3(3\psi_1+2\psi_2+\psi_4)}{\sqrt{2}}$	$-\frac{9\psi_1+3\psi_2+6\psi_4}{\sqrt{2}}$
22	$\frac{3(4\psi_1+5\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(5\psi_1+4\psi_2+2\psi_4)$	$\frac{3(\psi_1+2\psi_2+\psi_4)}{\sqrt{2}}$	$\frac{3(7\psi_1+5\psi_2+2\psi_4)}{\sqrt{2}}$
23	$-\frac{3(10\psi_1+3\psi_2+6\psi_3-3\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(3\psi_1+\psi_2+3\psi_3-\psi_4)$	$\frac{3(\psi_1-2\psi_2+2\psi_3-3\psi_4)}{\sqrt{2}}$	$\frac{9\psi_1+9\psi_2+6\psi_4}{\sqrt{2}}$
24	$\frac{3(2\psi_1+3\psi_2+2\psi_3+\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(\psi_1-\psi_2+3\psi_3-\psi_4)$	$-\frac{3(\psi_1-2\psi_2+2\psi_3-3\psi_4)}{\sqrt{2}}$	$-\frac{3(3\psi_1+3\psi_2+2\psi_4)}{\sqrt{2}}$
25	$\frac{3(2\psi_1-9\psi_2+6\psi_3-11\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(5\psi_1+\psi_2+3\psi_3-\psi_4)$	$-\frac{3(\psi_1-2\psi_2+2\psi_3-3\psi_4)}{\sqrt{2}}$	$-\frac{3(3\psi_1+3\psi_2+2\psi_4)}{\sqrt{2}}$
26	$\frac{3(20\psi_1+7\psi_2+8\psi_3-3\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1-\psi_2+2\psi_3-2\psi_4)$	$-\frac{3(15\psi_1+6\psi_2+6\psi_3-\psi_4)}{\sqrt{2}}$	$\frac{3(7\psi_1-3\psi_2+6\psi_3-6\psi_4)}{\sqrt{2}}$
27	$-3\sqrt{2}(4\psi_1+\psi_2+2\psi_3)$	$3\sqrt{2}(3\psi_1-4\psi_2+2\psi_3-4\psi_4)$	$-3\sqrt{2}(5\psi_1+\psi_2+3\psi_3)$	$-3\sqrt{2}(4\psi_1+\psi_2+3\psi_3-\psi_4)$
28	$-\frac{3(4\psi_1+5(\psi_2+\psi_4))}{\sqrt{2}}$	$-3\sqrt{2}(4\psi_1+\psi_2+2\psi_3)$	$-\frac{3(3\psi_1+8\psi_2+5\psi_4)}{\sqrt{2}}$	$\frac{3(5\psi_1+\psi_2)}{\sqrt{2}}$
29	$-\frac{3(2\psi_1-7\psi_2+2\psi_3-7\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(6\psi_1-\psi_2+4\psi_3-4\psi_4)$	$-\frac{3(7\psi_1-6\psi_2+4\psi_3-7\psi_4)}{\sqrt{2}}$	$\frac{3(5\psi_1-\psi_2+6\psi_3-2\psi_4)}{\sqrt{2}}$
30	$-\frac{6\psi_1+9\psi_2-6\psi_3+9\psi_4}{\sqrt{2}}$	$3\sqrt{2}(4\psi_1-\psi_2+2\psi_3-2\psi_4)$	$\frac{3(13\psi_1-2\psi_2+8\psi_3-9\psi_4)}{\sqrt{2}}$	$\frac{3(5\psi_1-5\psi_2+6\psi_3-6\psi_4)}{\sqrt{2}}$
31	$\frac{3(14\psi_1-5\psi_2+10\psi_3-9\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(6\psi_1+\psi_2+2\psi_3-2\psi_4)$	$-\frac{3(7\psi_1-2\psi_2+4\psi_3-3\psi_4)}{\sqrt{2}}$	$-\frac{3(15\psi_1+9\psi_2+6\psi_3+2\psi_4)}{\sqrt{2}}$
32	$-\frac{3(\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(2\psi_1-\psi_2+2\psi_3-3\psi_4)$	$-\frac{3(9\psi_1-2\psi_2+6\psi_3-5\psi_4)}{\sqrt{2}}$	$-\frac{3(3\psi_1+7\psi_2+4\psi_4)}{\sqrt{2}}$
33	$-\frac{3(8\psi_1+3\psi_2+6\psi_3-\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(3\psi_2-2\psi_3+3\psi_4)$	$\frac{3(5\psi_1-2\psi_2+6\psi_3-5\psi_4)}{\sqrt{2}}$	$-\frac{3(5\psi_1+5\psi_2+4\psi_4)}{\sqrt{2}}$
34	$-\frac{9(3\psi_2-2\psi_3+3\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_2+\psi_4)$	$\frac{3(3\psi_1-6\psi_2+6\psi_3-7\psi_4)}{\sqrt{2}}$	$\frac{3(9\psi_1+5\psi_2)}{\sqrt{2}}$
35	$-3\sqrt{2}(2\psi_1-\psi_2+2\psi_3-\psi_4)$	$-6\sqrt{2}(\psi_1+\psi_2+\psi_3)$	$3\sqrt{2}(\psi_1-2\psi_2+2\psi_3-3\psi_4)$	$-3\sqrt{2}(3\psi_1+3\psi_2+2\psi_4)$

0	5	6	7	8
1	$\frac{3(-8\psi_1+\psi_2-6\psi_3+3\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(4\psi_1-5\psi_2+4\psi_3-6\psi_4)$	$\frac{3(4\psi_1+\psi_2-2\psi_3-\psi_4)}{\sqrt{2}}$	$\frac{3(8\psi_1-5\psi_2+10\psi_3-9\psi_4)}{\sqrt{14}}$
2	$\frac{3(2\psi_1+3\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1-\psi_2+\psi_3-2\psi_4)$	$-\frac{3(3\psi_2+2\psi_3+\psi_4)}{\sqrt{2}}$	$-\frac{54\psi_1+45\psi_2-42\psi_3+57\psi_4}{\sqrt{14}}$
3	$\frac{3(12\psi_1+11\psi_2+7\psi_4)}{\sqrt{2}}$	$-9\sqrt{2}\psi_1$	$\frac{3(\psi_2+\psi_4)}{\sqrt{2}}$	$\frac{3(12\psi_1+7\psi_2+4\psi_3+5\psi_4)}{\sqrt{14}}$
4	$-3\sqrt{2}(\psi_1-5\psi_2+3\psi_3-5\psi_4)$	$3\sqrt{2}(3\psi_1+\psi_2+\psi_3+\psi_4)$	$-3\sqrt{2}(2\psi_1+\psi_2-2\psi_3+\psi_4)$	$3\sqrt{\frac{2}{7}}(8\psi_1+\psi_2+4\psi_3-4\psi_4)$
5	$\frac{6\psi_1-9(\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(2\psi_1-\psi_2+3\psi_3-3\psi_4)$	$\frac{3(4\psi_1+7\psi_2+3\psi_4)}{\sqrt{2}}$	$\frac{3(28\psi_1+9\psi_2+12\psi_3-3\psi_4)}{\sqrt{14}}$
6	$\frac{9(\psi_2+\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(\psi_1-3(\psi_2-\psi_3+\psi_4))$	$-\frac{3(6\psi_1+\psi_2+4\psi_3-3\psi_4)}{\sqrt{2}}$	$\frac{3(10\psi_1-9\psi_2+10\psi_3-15\psi_4)}{\sqrt{14}}$
7	$3\sqrt{2}(5\psi_1+\psi_2+3\psi_3-\psi_4)$	$3\sqrt{2}(2\psi_1-\psi_2+3\psi_3-\psi_4)$	$-3\sqrt{2}(3\psi_1-3\psi_2+3\psi_3-4\psi_4)$	$3\sqrt{\frac{2}{7}}(2\psi_1+3\psi_2+\psi_4)$
8	$-\frac{3(2\psi_1+3\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(3\psi_1+2\psi_2+\psi_4)$	$\frac{3(18\psi_1+5(\psi_2+2\psi_3-\psi_4))}{\sqrt{2}}$	$\frac{3(-3\psi_2-4\psi_3+\psi_4)}{\sqrt{14}}$
9	$-\frac{3(12\psi_1+5\psi_2+6\psi_3-\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(3\psi_1-3\psi_2+3\psi_3-4\psi_4)$	$-\frac{3(9\psi_2-4\psi_3+9\psi_4)}{\sqrt{2}}$	$\frac{3(4\psi_1+3\psi_2-4\psi_3+3\psi_4)}{\sqrt{14}}$
10	$\frac{3(2\psi_1+5\psi_2+3\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(2\psi_1+3\psi_2+2\psi_4)$	$-\frac{3(4\psi_1+\psi_2+\psi_4)}{\sqrt{2}}$	$\frac{3(-10\psi_1+3\psi_2-4\psi_3+3\psi_4)}{\sqrt{14}}$
11	$\frac{3(2\psi_1+\psi_2-\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(3\psi_1+3\psi_2+2\psi_4)$	$\frac{3(6\psi_1-\psi_2+2\psi_3-3\psi_4)}{\sqrt{2}}$	$\frac{78\psi_1-3\psi_2+36\psi_3-33\psi_4}{\sqrt{14}}$
12	$3\sqrt{2}(\psi_2+\psi_4)$	$3\sqrt{2}\psi_1$	$-3\sqrt{2}(\psi_1-\psi_2+\psi_3-2\psi_4)$	$3\sqrt{\frac{2}{7}}(8\psi_1+\psi_2+4\psi_3-4\psi_4)$
13	$\frac{-6\psi_1+9(\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(4\psi_1+5\psi_2+3\psi_4)$	$\frac{3(-2\psi_1+\psi_2+\psi_4)}{\sqrt{2}}$	$\frac{3(5\psi_2+2\psi_3+3\psi_4)}{\sqrt{14}}$
14	$\frac{3(18\psi_1+7\psi_2+6\psi_3-\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_2+\psi_4)$	$-\frac{9(2\psi_1+\psi_2+\psi_4)}{\sqrt{2}}$	$\frac{3(5\psi_2+2\psi_3+3\psi_4)}{\sqrt{14}}$
15	$-\frac{3(6\psi_1+3\psi_2+6\psi_3-\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(2\psi_1+\psi_2+\psi_4)$	$\frac{3(10\psi_1-\psi_2+4\psi_3-5\psi_4)}{\sqrt{2}}$	$-\frac{3(5\psi_2+2\psi_3+3\psi_4)}{\sqrt{14}}$
16	$\frac{3(8\psi_1-5\psi_2+6\psi_3-7\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_2+\psi_4)$	$\frac{3(-4\psi_1+\psi_2-4\psi_3+3\psi_4)}{\sqrt{2}}$	$3\sqrt{\frac{7}{2}}(2\psi_1-\psi_2+2\psi_3-\psi_4)$
17	$\frac{3(\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(6\psi_1+\psi_2+4\psi_3-2\psi_4)$	$-\frac{3(2\psi_1+9\psi_2-4\psi_3+7\psi_4)}{\sqrt{2}}$	$-\frac{3(4\psi_1+29\psi_2-16\psi_3+27\psi_4)}{\sqrt{14}}$
18	$-\frac{3(\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1+\psi_2)$	$\frac{3(16\psi_1+5\psi_2+10\psi_3-3\psi_4)}{\sqrt{2}}$	$\frac{3(24\psi_1+13\psi_2+16\psi_3+\psi_4)}{\sqrt{14}}$
19	$-3\sqrt{2}(\psi_2+\psi_4)$	$3\sqrt{2}(7\psi_1+4\psi_3-4\psi_4)$	$-3\sqrt{2}(3\psi_1-\psi_2+3\psi_3-2\psi_4)$	$3\sqrt{\frac{2}{7}}(2\psi_1-3\psi_2+6\psi_3-4\psi_4)$
20	$3\sqrt{2}(3\psi_1+\psi_2+\psi_3+\psi_4)$	$-3\sqrt{2}(\psi_2-\psi_3+\psi_4)$	$-3\sqrt{2}(\psi_1-\psi_2+\psi_3-2\psi_4)$	$3\sqrt{\frac{2}{7}}(-20\psi_1+5\psi_2-14\psi_3+11\psi_4)$
21	$-\frac{3(4\psi_1-7\psi_2+8\psi_3-11\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(\psi_1+2\psi_2+\psi_4)$	$\frac{3(8\psi_1+3\psi_2+2\psi_3-\psi_4)}{\sqrt{2}}$	$\frac{3(2\psi_1+3\psi_2+\psi_4)}{\sqrt{14}}$
22	$\frac{3(6\psi_1-5\psi_2+2\psi_3-5\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(\psi_1+\psi_2+\psi_3)$	$-\frac{3(6\psi_1+5\psi_2+3\psi_4)}{\sqrt{2}}$	$-\frac{3(2\psi_1+17\psi_2+15\psi_4)}{\sqrt{14}}$
23	$\frac{3(2\psi_1+3\psi_2+\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(5\psi_1+\psi_2+\psi_3-\psi_4)$	$-\frac{3(12\psi_1+5\psi_2+4\psi_3-\psi_4)}{\sqrt{2}}$	$\frac{48\psi_1+45\psi_2+6\psi_3+33\psi_4}{\sqrt{14}}$
24	$\frac{3(2\psi_1+5\psi_2+3\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(-5\psi_1+\psi_2-3\psi_3+3\psi_4)$	$\frac{3(8\psi_1+9\psi_2+7\psi_4)}{\sqrt{2}}$	$\frac{3(-8\psi_1+\psi_2-6\psi_3+\psi_4)}{\sqrt{14}}$
25	$\frac{3(2\psi_1+\psi_2-\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(3\psi_1-3\psi_2+\psi_3-3\psi_4)$	$\frac{3(4\psi_1+\psi_2-\psi_4)}{\sqrt{2}}$	$-\frac{3(48\psi_1+19\psi_2+18\psi_3-5\psi_4)}{\sqrt{14}}$
26	$-\frac{3(2\psi_1+3\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}\psi_1$	$-\frac{3(6\psi_1-3\psi_2+4\psi_3-5\psi_4)}{\sqrt{2}}$	$\frac{6\psi_1-81\psi_2+48\psi_3-93\psi_4}{\sqrt{14}}$
27	$3\sqrt{2}(5\psi_1-\psi_2+3\psi_3-3\psi_4)$	$3\sqrt{2}(5\psi_1+\psi_2+3\psi_3-\psi_4)$	$-3\sqrt{2}(\psi_2+\psi_4)$	$-3\sqrt{\frac{2}{7}}(6\psi_1+3\psi_2+6\psi_3-2\psi_4)$
28	$\frac{3(12\psi_1+\psi_2+6\psi_3-5\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(4\psi_1+\psi_2+3\psi_3-\psi_4)$	$\frac{3(6\psi_1-\psi_2+4\psi_3-3\psi_4)}{\sqrt{2}}$	$-\frac{3(30\psi_1+15\psi_2+16\psi_3-3\psi_4)}{\sqrt{14}}$
29	$\frac{3(\psi_2+\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(7\psi_1+3(\psi_2+\psi_3))$	$\frac{6\psi_1+9\psi_2+6\psi_3+9\psi_4}{\sqrt{2}}$	$\frac{36\psi_1+39\psi_2-6\psi_3+9\psi_4}{\sqrt{14}}$
30	$\frac{3(\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(5\psi_1+3(\psi_2+\psi_3))$	$-\frac{42\psi_1+9\psi_2-30\psi_3+21\psi_4}{\sqrt{2}}$	$\frac{3(4\psi_1+\psi_2-2\psi_3-\psi_4)}{\sqrt{14}}$
31	$\frac{3(\psi_2+\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(3\psi_1-\psi_2+3\psi_3-2\psi_4)$	$\frac{6\psi_1-3(\psi_2-2\psi_3+\psi_4)}{\sqrt{2}}$	$-\frac{3(12\psi_1+7\psi_2+5(-2\psi_3+\psi_4))}{\sqrt{14}}$
32	$\frac{3(6\psi_1-5\psi_2+2\psi_3-5\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(2\psi_1+\psi_2)$	$-\frac{6\psi_1+9\psi_2-6\psi_3+9\psi_4}{\sqrt{2}}$	$-\frac{3(10\psi_1-5\psi_2+6\psi_3-7\psi_4)}{\sqrt{14}}$
33	$\frac{3(10\psi_1+\psi_2+2\psi_3-3\psi_4)}{\sqrt{2}}$	$-3\sqrt{2}(2\psi_1+\psi_2)$	$\frac{6\psi_1-9\psi_2+6\psi_3-9\psi_4}{\sqrt{2}}$	$\frac{3(10\psi_1-\psi_2+2\psi_3+\psi_4)}{\sqrt{14}}$
34	$-\frac{3(14\psi_1+\psi_2+10\psi_3-7\psi_4)}{\sqrt{2}}$	$3\sqrt{2}(2\psi_1+\psi_2)$	$-\frac{6\psi_1+9\psi_2-6\psi_3+9\psi_4}{\sqrt{2}}$	$-\frac{9(14\psi_1+\psi_2+6\psi_3-5\psi_4)}{\sqrt{14}}$
35	$-3\sqrt{2}(\psi_2+\psi_4)$	$6\sqrt{2}(2\psi_1+\psi_2)$	$3\sqrt{2}(2\psi_1-3\psi_2+2\psi_3-3\psi_4)$	$-3\sqrt{\frac{2}{7}}(5\psi_2+2\psi_3+3\psi_4)$

0	9	10	11	12
1	$-3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$	$\frac{-51\psi_1 + 30\psi_2 - 54\psi_3 + 51\psi_4}{\sqrt{14}}$	$-3\sqrt{\frac{7}{2}}(\psi_1 + \psi_2)$	$\frac{3(2\psi_1 - 5\psi_2 - 6\psi_3 - \psi_4)}{\sqrt{14}}$
2	$-3\sqrt{\frac{2}{7}}(8\psi_1 + \psi_2 + 4\psi_3 - 4\psi_4)$	$\frac{3(13\psi_1 - 6\psi_2 + 8\psi_3 - 13\psi_4)}{\sqrt{14}}$	$-\frac{3(15\psi_1 + 13\psi_2 + 6(\psi_3 + \psi_4))}{\sqrt{14}}$	$-3\sqrt{\frac{7}{2}}(\psi_2 + \psi_4)$
3	$3\sqrt{\frac{2}{7}}(\psi_1 + 5\psi_2 + 4\psi_4)$	$-\frac{3(7\psi_1 - 6\psi_2 + 6\psi_3 - 5\psi_4)}{\sqrt{14}}$	$-\frac{3(13\psi_1 + 3\psi_2 + 6\psi_3 - 2\psi_4)}{\sqrt{14}}$	$\frac{3(50\psi_1 + \psi_2 + 32\psi_3 - 25\psi_4)}{\sqrt{14}}$
4	$-3\sqrt{\frac{2}{7}}(5\psi_1 + 4\psi_3 - 2\psi_4)$	$3\sqrt{\frac{2}{7}}(9\psi_1 + 7\psi_2 + 3\psi_3 + 2\psi_4)$	$3\sqrt{\frac{2}{7}}(-6\psi_1 + \psi_2 - 3\psi_3 + 3\psi_4)$	$3\sqrt{\frac{2}{7}}(17\psi_1 + 3\psi_2 + 5\psi_3 - 5\psi_4)$
5	$-3\sqrt{\frac{2}{7}}(4\psi_1 - 5\psi_2 + 4\psi_3 - 6\psi_4)$	$-\frac{3(25\psi_1 + 8\psi_2 + 12\psi_3 - \psi_4)}{\sqrt{14}}$	$-\frac{3(\psi_1 + 5\psi_2 + 4\psi_4)}{\sqrt{14}}$	$\frac{-60\psi_1 + 45\psi_2 - 30\psi_3 + 51\psi_4}{\sqrt{14}}$
6	$-3\sqrt{\frac{2}{7}}(9\psi_1 + \psi_2 + 9\psi_3 - 3\psi_4)$	$3\sqrt{\frac{7}{2}}(\psi_1 - 2\psi_2 + 2\psi_3 - 3\psi_4)$	$3\sqrt{\frac{7}{2}}(3\psi_1 + 3\psi_2 + 2\psi_4)$	$\frac{9(2\psi_1 + 3\psi_2 + \psi_4)}{\sqrt{14}}$
7	$-3\sqrt{\frac{2}{7}}(5\psi_1 + 7\psi_2 - 3\psi_3 + 5\psi_4)$	$-3\sqrt{\frac{2}{7}}(\psi_1 - 8\psi_2 + 6\psi_3 - 8\psi_4)$	$3\sqrt{14}(2\psi_1 + \psi_2)$	$-3\sqrt{\frac{2}{7}}(\psi_1 + \psi_2 - 3\psi_3 + 3\psi_4)$
8	$3\sqrt{\frac{2}{7}}(6\psi_1 - \psi_2 + 3\psi_3 - 3\psi_4)$	$\frac{3(9\psi_1 - 2\psi_2 + 12\psi_3 - 9\psi_4)}{\sqrt{14}}$	$3\sqrt{\frac{7}{2}}(3\psi_1 + 3\psi_2 + 2\psi_4)$	$3\sqrt{\frac{7}{2}}(\psi_2 + \psi_4)$
9	$3\sqrt{\frac{2}{7}}(\psi_1 - 8\psi_2 + 6\psi_3 - 8\psi_4)$	$\frac{3(23\psi_1 - 2\psi_2 + 12\psi_3 - 9\psi_4)}{\sqrt{14}}$	$-3\sqrt{\frac{7}{2}}(\psi_1 + 3\psi_2 + 2\psi_4)$	$\frac{3(2\psi_1 + 9\psi_2 - 6\psi_3 + 13\psi_4)}{\sqrt{14}}$
10	$3\sqrt{\frac{2}{7}}(15\psi_1 + 6\psi_2 + 6\psi_3 - \psi_4)$	$\frac{-93\psi_1 + 30\psi_2 - 54\psi_3 + 51\psi_4}{\sqrt{14}}$	$\frac{3(45\psi_1 + 25\psi_2 + 18\psi_3 + 4\psi_4)}{\sqrt{14}}$	$\frac{3(4\psi_1 - \psi_2 - 5\psi_4)}{\sqrt{14}}$
11	$-3\sqrt{\frac{2}{7}}(9\psi_1 + 7\psi_2 + 3\psi_3 + 2\psi_4)$	$\frac{3(-39\psi_1 + 4\psi_2 - 24\psi_3 + 25\psi_4)}{\sqrt{14}}$	$\frac{75\psi_1 - 33\psi_2 + 72\psi_3 - 54\psi_4}{\sqrt{14}}$	$-\frac{3(4\psi_1 + 13\psi_2 + 9\psi_4)}{\sqrt{14}}$
12	$3\sqrt{\frac{2}{7}}(6\psi_1 - \psi_2 + 3\psi_3 - 3\psi_4)$	$-3\sqrt{\frac{2}{7}}(9\psi_1 - 9\psi_2 + 5\psi_3 - 9\psi_4)$	$-3\sqrt{\frac{2}{7}}(5\psi_1 - 5\psi_2 + 9\psi_3 - 5\psi_4)$	$3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$
13	$3\sqrt{\frac{2}{7}}(5\psi_1 + \psi_2 + 3\psi_3)$	$-\frac{3(5\psi_1 + 18\psi_2 + 13\psi_4)}{\sqrt{14}}$	$-\frac{3(5\psi_1 + 3\psi_2 - 6\psi_3 + 4\psi_4)}{\sqrt{14}}$	$\frac{60\psi_1 - 45\psi_2 + 30\psi_3 - 51\psi_4}{\sqrt{14}}$
14	$3\sqrt{\frac{2}{7}}(5\psi_1 + \psi_2 + 3\psi_3)$	$\frac{57\psi_1 + 9(2\psi_2 + 4\psi_3 + \psi_4)}{\sqrt{14}}$	$-\frac{3(21\psi_1 + 3\psi_2 + 18\psi_3 - 8\psi_4)}{\sqrt{14}}$	$\frac{48\psi_1 - 57\psi_2 + 66\psi_3 - 87\psi_4}{\sqrt{14}}$
15	$-3\sqrt{\frac{2}{7}}(5\psi_1 + \psi_2 + 3\psi_3)$	$-\frac{3(31\psi_1 + 10\psi_2 + 12\psi_3 - 5\psi_4)}{\sqrt{14}}$	$\frac{3(-19\psi_1 + 7\psi_2 - 18\psi_3 + 16\psi_4)}{\sqrt{14}}$	$\frac{60\psi_1 - 3\psi_2 + 30\psi_3 - 9\psi_4}{\sqrt{14}}$
16	$-3\sqrt{14}(\psi_1 + \psi_2 + \psi_3)$	$-3\sqrt{\frac{7}{2}}(\psi_1 + 2\psi_2 + \psi_4)$	$-\frac{3(25\psi_1 + 3\psi_2 + 10\psi_3 - 4\psi_4)}{\sqrt{14}}$	$-\frac{3(10\psi_1 + 7\psi_2 - 6\psi_3 + 3\psi_4)}{\sqrt{14}}$
17	$-3\sqrt{\frac{2}{7}}(\psi_1 - 2\psi_2 - 3\psi_4)$	$-\frac{3(13\psi_1 + 10\psi_2 + 6\psi_3 + 5\psi_4)}{\sqrt{14}}$	$\frac{3(13\psi_1 - 7\psi_2 + 2\psi_3 - 8\psi_4)}{\sqrt{14}}$	$\frac{3(2\psi_1 + 3\psi_2 + \psi_4)}{\sqrt{14}}$
18	$-3\sqrt{\frac{2}{7}}(\psi_1 - 9\psi_2 + 7\psi_3 - 10\psi_4)$	$\frac{3(-3\psi_1 + 4\psi_2 - 12\psi_3 + 5\psi_4)}{\sqrt{14}}$	$\frac{3(11\psi_1 - 9\psi_2 + 8\psi_3 - 14\psi_4)}{\sqrt{14}}$	$-\frac{3(2\psi_1 + 3\psi_2 + \psi_4)}{\sqrt{14}}$
19	$3\sqrt{\frac{2}{7}}(-6\psi_1 + 5\psi_2 - 7\psi_3 + 11\psi_4)$	$-3\sqrt{\frac{2}{7}}(5\psi_1 + 7\psi_2 - 3\psi_3 + 5\psi_4)$	$3\sqrt{\frac{2}{7}}(\psi_1 + \psi_2 - 3\psi_3 + 3\psi_4)$	$-3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$
20	$3\sqrt{\frac{2}{7}}(17\psi_1 + 3\psi_2 + 5\psi_3 - 5\psi_4)$	$-3\sqrt{\frac{2}{7}}(5\psi_1 + 4\psi_3 - 2\psi_4)$	$3\sqrt{\frac{2}{7}}(8\psi_1 + \psi_2 + 4\psi_3 - 4\psi_4)$	$3\sqrt{\frac{2}{7}}(13\psi_1 + 7\psi_2 + 9\psi_3 - \psi_4)$
21	$-9\sqrt{\frac{2}{7}}(6\psi_1 - \psi_2 + 3\psi_3 - 3\psi_4)$	$\frac{3(-11\psi_1 + 2\psi_2 - 8\psi_3 + 7\psi_4)}{\sqrt{14}}$	$\frac{3(-5\psi_1 + 11\psi_2 - 8\psi_3 + 10\psi_4)}{\sqrt{14}}$	$\frac{3(58\psi_1 + 21\psi_2 + 24\psi_3 - 5\psi_4)}{\sqrt{14}}$
22	$3\sqrt{\frac{2}{7}}(-17\psi_1 + 4(\psi_2 - 3\psi_3 + 3\psi_4))$	$\frac{3(\psi_1 - 2\psi_2 - 3\psi_4)}{\sqrt{14}}$	$\frac{3(21\psi_1 - 9\psi_2 + 16\psi_3 - 18\psi_4)}{\sqrt{14}}$	$\frac{3(12\psi_1 - 7\psi_2 + 18\psi_3 - 9\psi_4)}{\sqrt{14}}$
23	$-3\sqrt{\frac{2}{7}}(11\psi_1 + \psi_2 + 5\psi_3 - \psi_4)$	$-\frac{3(15\psi_1 + 6\psi_2 + 6\psi_3 - \psi_4)}{\sqrt{14}}$	$\frac{3(13\psi_1 + \psi_2 + 8\psi_3 - 6\psi_4)}{\sqrt{14}}$	$-3\sqrt{\frac{7}{2}}(\psi_2 + \psi_4)$
24	$-3\sqrt{\frac{2}{7}}(9\psi_1 - 9\psi_2 + 5\psi_3 - 9\psi_4)$	$\frac{3(15\psi_1 + 6\psi_2 + 6\psi_3 - \psi_4)}{\sqrt{14}}$	$-\frac{3(13\psi_1 + \psi_2 + 8\psi_3 - 6\psi_4)}{\sqrt{14}}$	$\frac{3(4\psi_1 - \psi_2 - 5\psi_4)}{\sqrt{14}}$
25	$3\sqrt{\frac{2}{7}}(\psi_1 + \psi_2 - 3\psi_3 + 3\psi_4)$	$\frac{3(15\psi_1 + 6\psi_2 + 6\psi_3 - \psi_4)}{\sqrt{14}}$	$-\frac{3(13\psi_1 + \psi_2 + 8\psi_3 - 6\psi_4)}{\sqrt{14}}$	$-\frac{3(4\psi_1 + 13\psi_2 + 9\psi_4)}{\sqrt{14}}$
26	$3\sqrt{\frac{2}{7}}(13\psi_1 + 3\psi_2 + 6\psi_3 - 2\psi_4)$	$-3\sqrt{\frac{7}{2}}(\psi_1 - 2\psi_2 + 2\psi_3 - 3\psi_4)$	$-\frac{3(19\psi_1 + 9\psi_2 + 2(\psi_3 + \psi_4))}{\sqrt{14}}$	$3\sqrt{\frac{7}{2}}(\psi_2 + \psi_4)$
27	$-3\sqrt{\frac{2}{7}}(\psi_1 + 4\psi_2 - 6\psi_3 + 2\psi_4)$	$-3\sqrt{14}(\psi_1 + \psi_2 + \psi_3)$	$3\sqrt{\frac{2}{7}}(6\psi_1 + \psi_2 + \psi_3 + \psi_4)$	$-3\sqrt{\frac{2}{7}}(5\psi_1 + 7\psi_2 - 3\psi_3 + 5\psi_4)$
28	$-3\sqrt{\frac{2}{7}}(6\psi_1 + 3\psi_2 + 6\psi_3 - 2\psi_4)$	$3\sqrt{\frac{7}{2}}(-\psi_1 + \psi_4)$	$\frac{3(19\psi_1 - 5\psi_2 + 16\psi_3 - 12\psi_4)}{\sqrt{14}}$	$-\frac{3(10\psi_1 + 21\psi_2 - 6\psi_3 + 17\psi_4)}{\sqrt{14}}$
29	$3\sqrt{\frac{2}{7}}(12\psi_1 + 11\psi_2 + 6\psi_4)$	$\frac{3(\psi_1 + 10\psi_2 - 12\psi_3 + 7\psi_4)}{\sqrt{14}}$	$\frac{3(-9\psi_1 + 13\psi_2 - 2\psi_3 + 10\psi_4)}{\sqrt{14}}$	$\frac{3(2\psi_1 + 3\psi_2 + \psi_4)}{\sqrt{14}}$
30	$3\sqrt{\frac{2}{7}}(2\psi_1 - 3\psi_2 + 6\psi_3 - 4\psi_4)$	$-\frac{3(27\psi_1 + 30\psi_2 + 17\psi_4)}{\sqrt{14}}$	$\frac{3(-17\psi_1 + \psi_2 - 2\psi_3 + 6\psi_4)}{\sqrt{14}}$	$\frac{3(2\psi_1 + 3\psi_2 + \psi_4)}{\sqrt{14}}$
31	$-3\sqrt{\frac{2}{7}}(13\psi_2 - 6\psi_3 + 12\psi_4)$	$-\frac{3(7\psi_1 + 2\psi_2 + 12\psi_3 - 3\psi_4)}{\sqrt{14}}$	$\frac{3(-13\psi_1 + 5\psi_2 - 14\psi_3 + 18\psi_4)}{\sqrt{14}}$	$\frac{3(2\psi_1 + 3\psi_2 + \psi_4)}{\sqrt{14}}$
32	$-3\sqrt{\frac{2}{7}}(16\psi_1 + 11\psi_2 + 6\psi_3 + 3\psi_4)$	$\frac{3(7\psi_1 + 6\psi_2 - 6\psi_3 + 5\psi_4)}{\sqrt{14}}$	$\frac{3(-5\psi_1 + 3\psi_2 + 8\psi_4)}{\sqrt{14}}$	$\frac{3(12\psi_1 - 7\psi_2 + 18\psi_3 - 9\psi_4)}{\sqrt{14}}$
33	$3\sqrt{\frac{2}{7}}(14\psi_1 + \psi_2 + 6\psi_3 - 5\psi_4)$	$-\frac{3(27\psi_1 + 6\psi_2 + 10\psi_3 - 3\psi_4)}{\sqrt{14}}$	$-\frac{3(27\psi_1 + 7\psi_2 + 16\psi_3 - 8\psi_4)}{\sqrt{14}}$	$\frac{36\psi_1 - 63\psi_2 + 54\psi_3 - 69\psi_4}{\sqrt{14}}$
34	$-3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$	$-\frac{3(29\psi_1 + 2\psi_2 + 10\psi_3 - 9\psi_4)}{\sqrt{14}}$	$\frac{45\psi_1 - 75\psi_2 + 48\psi_3 - 84\psi_4}{\sqrt{14}}$	$\frac{3(32\psi_1 + 21\psi_2 + 6\psi_3 + 11\psi_4)}{\sqrt{14}}$
35	$6\sqrt{\frac{2}{7}}(5\psi_1 + \psi_2 + 3\psi_3)$	$-3\sqrt{\frac{2}{7}}(15\psi_1 + 6\psi_2 + 6\psi_3 - \psi_4)$	$-3\sqrt{\frac{2}{7}}(13\psi_1 + \psi_2 + 8\psi_3 - 6\psi_4)$	$-3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$

0	13	14
1	$-3\sqrt{14}(2\psi_1 + \psi_2)$	$3\sqrt{\frac{7}{2}}(2\psi_1 - 3\psi_2 + 2\psi_3 - 3\psi_4)$
2	$3\sqrt{\frac{2}{7}}(9\psi_1 + 7\psi_2 + 3\psi_3 + 2\psi_4)$	$\frac{3(18\psi_1 - \psi_2 + 14\psi_3 - 5\psi_4)}{\sqrt{14}}$
3	$-9\sqrt{\frac{2}{7}}(5\psi_1 + 4\psi_3 - 2\psi_4)$	$\frac{3(2\psi_1 + 3\psi_2 + \psi_4)}{\sqrt{14}}$
4	$3\sqrt{\frac{2}{7}}(13\psi_1 + 7\psi_2 + 9\psi_3 - \psi_4)$	$3\sqrt{\frac{2}{7}}(-20\psi_1 + 5\psi_2 - 14\psi_3 + 11\psi_4)$
5	$3\sqrt{\frac{2}{7}}(20\psi_1 + 7\psi_2 + 9\psi_3 - \psi_4)$	$\frac{6\psi_1 - 33\psi_2 - 39\psi_4}{\sqrt{14}}$
6	$3\sqrt{\frac{2}{7}}(-13\psi_1 + 3\psi_2 - 5\psi_3 + 7\psi_4)$	$\frac{3(16\psi_1 + 13\psi_2 + 4\psi_3 + 7\psi_4)}{\sqrt{14}}$
7	$-3\sqrt{\frac{2}{7}}(4\psi_1 - 9\psi_2 + \psi_3 - 7\psi_4)$	$3\sqrt{\frac{2}{7}}(11\psi_1 + 5\psi_2 + \psi_3)$
8	$3\sqrt{\frac{2}{7}}(-11\psi_1 + 2\psi_2 - 8\psi_3 + 7\psi_4)$	$-\frac{3(20\psi_1 + 25\psi_2 - 2\psi_3 + 21\psi_4)}{\sqrt{14}}$
9	$-3\sqrt{\frac{2}{7}}(11\psi_1 + 5\psi_2 + \psi_3)$	$-\frac{3(34\psi_1 + 11\psi_2 + 12\psi_3 - 7\psi_4)}{\sqrt{14}}$
10	$3\sqrt{\frac{2}{7}}(8\psi_1 + \psi_2 + 4\psi_3 - 4\psi_4)$	$-\frac{3(22\psi_1 + 3\psi_2 + 16\psi_3 - 7\psi_4)}{\sqrt{14}}$
11	$-3\sqrt{\frac{2}{7}}(13\psi_1 + \psi_2 + 8\psi_3 - 6\psi_4)$	$\frac{3(4\psi_1 - 11\psi_2 + 10\psi_3 - 11\psi_4)}{\sqrt{14}}$
12	$3\sqrt{\frac{2}{7}}(5\psi_1 + 4\psi_3 - 2\psi_4)$	$3\sqrt{\frac{2}{7}}(9\psi_1 + 7\psi_2 + 3\psi_3 + 2\psi_4)$
13	$3\sqrt{\frac{2}{7}}(-14\psi_1 + \psi_2 - 8\psi_3 + 9\psi_4)$	$-\frac{3(8\psi_1 - 3\psi_2 + 8\psi_3 - 5\psi_4)}{\sqrt{14}}$
14	$-3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$	$-\frac{9(12\psi_1 + 3\psi_2 + 8\psi_3 - 3\psi_4)}{\sqrt{14}}$
15	$3\sqrt{\frac{2}{7}}(12\psi_1 + 3\psi_2 + 8\psi_3 - 3\psi_4)$	$\frac{-57\psi_2 + 36\psi_3 - 51\psi_4}{\sqrt{14}}$
16	$-3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$	$\frac{3(14\psi_1 + 3\psi_2 + 4\psi_3 - \psi_4)}{\sqrt{14}}$
17	$3\sqrt{\frac{2}{7}}(8\psi_1 + 5\psi_2 + 4\psi_4)$	$\frac{3(-28\psi_1 + 5\psi_2 - 12\psi_3 + 17\psi_4)}{\sqrt{14}}$
18	$3\sqrt{\frac{2}{7}}(\psi_1 + 5\psi_2 + 4\psi_4)$	$-\frac{3(14\psi_1 + 9\psi_2 - 2\psi_3 + 11\psi_4)}{\sqrt{14}}$
19	$-3\sqrt{\frac{2}{7}}(13\psi_1 + 16\psi_2 + 10\psi_4)$	$3\sqrt{\frac{2}{7}}(7\psi_1 - \psi_2 + \psi_3 - 2\psi_4)$
20	$-3\sqrt{\frac{2}{7}}(6\psi_1 - \psi_2 + 3\psi_3 - 3\psi_4)$	$3\sqrt{\frac{2}{7}}(9\psi_1 + 7\psi_2 + 3\psi_3 + 2\psi_4)$
21	$3\sqrt{\frac{2}{7}}(\psi_1 - 2\psi_2 - 3\psi_4)$	$\frac{3(6\psi_1 - 15\psi_2 + 10\psi_3 - 17\psi_4)}{\sqrt{14}}$
22	$3\sqrt{\frac{2}{7}}(5\psi_1 + \psi_2 + 3\psi_3)$	$\frac{3(-24\psi_1 + \psi_2 - 16\psi_3 + 13\psi_4)}{\sqrt{14}}$
23	$3\sqrt{\frac{2}{7}}(7\psi_1 - 9\psi_2 + 9\psi_3 - 11\psi_4)$	$\frac{3(-6\psi_1 + 17\psi_2 - 12\psi_3 + 21\psi_4)}{\sqrt{14}}$
24	$3\sqrt{\frac{2}{7}}(5\psi_1 + 7\psi_2 - 3\psi_3 + 5\psi_4)$	$\frac{3(42\psi_1 + 11\psi_2 + 24\psi_3 - 13\psi_4)}{\sqrt{14}}$
25	$3\sqrt{\frac{2}{7}}(5\psi_1 - 5\psi_2 + 9\psi_3 - 5\psi_4)$	$\frac{3(6\psi_1 - 13\psi_2 + 8\psi_3 - 13\psi_4)}{\sqrt{14}}$
26	$-3\sqrt{\frac{2}{7}}(5\psi_1 + 4\psi_3 - 2\psi_4)$	$\frac{3(8\psi_1 + 9\psi_2 - 4\psi_3 + 5\psi_4)}{\sqrt{14}}$
27	$-3\sqrt{\frac{2}{7}}(\psi_1 + \psi_2 - 3\psi_3 + 3\psi_4)$	$-3\sqrt{\frac{2}{7}}(2\psi_1 + 3\psi_2 + \psi_4)$
28	$3\sqrt{\frac{2}{7}}(6\psi_1 + \psi_2 + \psi_3 + \psi_4)$	$-\frac{3(4\psi_1 + 3\psi_2 - 4\psi_3 + 3\psi_4)}{\sqrt{14}}$
29	$3\sqrt{\frac{2}{7}}(5\psi_1 - 3\psi_2 + 7\psi_3 - 8\psi_4)$	$\frac{3(8\psi_1 + 17\psi_2 + 2\psi_3 + 7\psi_4)}{\sqrt{14}}$
30	$3\sqrt{\frac{2}{7}}(5\psi_1 + 3\psi_2 + \psi_3 + 4\psi_4)$	$\frac{3(8\psi_1 + \psi_2 - 10\psi_3 + 3\psi_4)}{\sqrt{14}}$
31	$3\sqrt{\frac{2}{7}}(7\psi_1 - \psi_2 + \psi_3 - 2\psi_4)$	$\frac{3(5\psi_2 + 2\psi_3 + 3\psi_4)}{\sqrt{14}}$
32	$3\sqrt{\frac{2}{7}}(4\psi_1 - 5\psi_2 + 4\psi_3 - 6\psi_4)$	$\frac{3(4\psi_1 + \psi_2 - 2\psi_3 - \psi_4)}{\sqrt{14}}$
33	$-3\sqrt{\frac{2}{7}}(4\psi_1 - 5\psi_2 + 4\psi_3 - 6\psi_4)$	$\frac{3(-4\psi_1 - \psi_2 + 2\psi_3 + \psi_4)}{\sqrt{14}}$
34	$3\sqrt{\frac{2}{7}}(4\psi_1 - 5\psi_2 + 4\psi_3 - 6\psi_4)$	$\frac{3(4\psi_1 + \psi_2 - 2\psi_3 - \psi_4)}{\sqrt{14}}$
35	$6\sqrt{\frac{2}{7}}(4\psi_1 - 5\psi_2 + 4\psi_3 - 6\psi_4)$	$-3\sqrt{\frac{2}{7}}(4\psi_1 + \psi_2 - 2\psi_3 - \psi_4)$

References

- [1] Klein, F. (1878). *Ueber die Transformation siebenter Ordnung der elliptischen Functionen* Mathematische Annalen 14 (3): 428471.
- [2] Hurwitz, A. (1893), *Über algebraische Gebilde mit Eindeutigen Transformationen in sich*, Mathematische Annalen 41 (3): 403442
- [3] Fano, G. (1892), *Sui postulati fondamentali della geometria proiettiva*, Giornale di Matematiche 30: 106132
- [4] R. C. King, F. Toumazet and B. G. Wybourne, *A finite subgroup of the exceptional Lie group G_2* , J. Phys. A: Math. Gen. 32 (1999) 85278537.
- [5] P.G.O. Freund and M.A. Rubin, *Dynamics of Dimensional reduction* Phys. Lett. **B97** (1980) 233
- [6] M. A. Awada, M. J. Duff and C. N. Pope, *$N = 8$ Supergravity Breaks Down To $N = 1$* Phys. Rev. Lett. **50** (1983) 294.
- [7] L. Castellani, R. D'Auria, P. Fré *$SU(3) \times SU(2) \times U(1)$ from $d=11$ supergravity* Nucl. Phys. **B239** (1984) 610.
- [8] F. Englert, *Spontaneous Compactification of Eleven-Dimensional Supergravity*, Phys. Lett. B **119** (1982) 339. doi:10.1016/0370-2693(82)90684-0
- [9] P. Fre' and M. Trigiante, *Twisted tori and fluxes: A No go theorem for Lie groups of weak $G(2)$ holonomy*, Nucl. Phys. B **751** (2006) 343 doi:10.1016/j.nuclphysb.2006.06.006 [hep-th/0603011].
- [10] A. Bilal, J.P. Derendinger, K. Sfetsos, *(Weak) G_2 Holonomy from self duality, flux and supersymmetry* Nucl. Phys. **B628** (2002) 112, ArXiv hep-th 011274
- [11] R. DAuria, P. Fré *On the Fermion Mass Spectrum of Kaluza Klein Supergravity* Ann. of Physics 157 , 1, (1984)
- [12] P. Fre', P. A. Grassi, L. Ravera and M. Trigiante, *Minimal $d = 7$ Supergravity and the supersymmetry of Arnold-Beltrami Flux branes*, arXiv:1511.06245 [hep-th].
- [13] E. Beltrami, *Opere matematiche*, **4** (1889) 304.
- [14] Hénon, M., *Sur la topologie des lignes de courant dans un cas particulier*, C. R. Acad. Sci. Paris **262** (1966) 312314.
- [15] J. Etnyre, R. Ghrist, *Contact topology and hydrodynamics: I. Beltrami fields and the Seifert conjecture*, Nonlinearity **13(2)** (2000) 441-458.
R. Ghrist, *On the contact geometry and topology of ideal fluids*, in *Handbook of Mathematical Fluid Dynamics*, Vol. **IV** (2007) 1 - 38.
- [16] V.I. Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier **16** (1966) 316 - 361.
- [17] V. I. Arnold, B. A. Khesin, *Topological Methods in Hydrodynamics*, Springer-Verlag, 1998.

- [18] The ABC flows have been discovered by Gromeka in 1881, rediscovered by Beltrami [13], and proposed for study in hydrodynamics in [16] and in magnetohydrodynamics in:
 S. Childress, *Construction of steady-state hydromagnetic dynamos. I. Spatially periodic fields*, Report MF-53, Courant Inst. of Math. Sci. (1967); *New solutions of the kinematic dynamo problem*, J. Math. Phys. **11** (1970) 3063 - 3076.
 Further important contributions on ABC flows are contained in the following short list of papers:
 T. Dombre, U. Frisch, J.M. Greene, M. Henon, A. Mehr, A.M. Soward, *Chaotic streamlines in the ABC flows*, J. Fluid Mech. **167** (1986) 353 - 391.
 O. I. Bogoyavlenskij, *Infinite families of exact periodic solutions to the Navier-Stokes equations*, Moscow Mathematical Journal **3** (2003) 1 - 10; O. Bogoyavlenskij, B. Fuchssteiner, *Exact NSE solutions with crystallographic symmetries and no transfer of energy through the spectrum*, J. Geom. Phys. **54** (2005) 324 - 338.
 V. I. Arnold, *On the evolution of a magnetic field under the action of transport and diffusion*, in *VLADIMIR I. ARNOLD: Collected Works, VOLUME II, Hydrodynamics, Bifurcation Theory, and Algebraic Geometry 1965-1972* (Edited by Alexander B. Givental, Boris A. Khesin, Alexander N. Varchenko, Victor A. Vassiliev, Oleg Ya. Viro), 405 - 419, Springer-Verlag Berlin Heidelberg 2014. Originally published in: *Some Problems in Modern Analysis*, 8-21 (Russian), Izd. MGU, Moscow 1984.
 G. E. Marsh, *Force-Free Magnetic Fields: Solutions, Topology and Applications*, World Scientific (Singapore), 1996.
 S. Childress, A. D. Gilbert, *Stretch, twist, fold: the fast dynamo*, Springer-Verlag, 1995.
 S. E. Jones, A. D. Gilbert, *Dynamo action in the ABC flows using symmetries*, Geophys. Astrophys. Fluid Dyn. **108** (2014) 83 - 116.
- [19] P. Fré and A. Sorin *Classification of Arnold Beltrami Flows and their Hidden Symmetries* Physics of Particles and Nuclei, 2015, Vol 46, No 4, pp 497-632
 arXiv:math-ph 1501.04604.
- [20] P. Fre and A. S. Sorin, *2-branes with Arnold-Beltrami Fluxes from Minimal d=7 Supergravity*, Fortsch. Phys. **63** (2015) 411 [arXiv:1504.06802 [hep-th]].
- [21] P. Fre, P. A. Grassi and A. S. Sorin, *Hyperinstantons, the Beltrami Equation, and Triholomorphic Maps*, arXiv:1509.09056 [hep-th].
- [22] P.K. Townsend and P. van Nieuwenhuizen *Gauged Seven Dimensional Supergravity* Phys. Lett. **B125** (1983), 41
- [23] Abdus Salm and E. Sezgin *SO(4) Gauging of $\mathcal{N} = 2$ supergravity in seven dimensions* Phys. Lett. **B126** (1983), 295
- [24] E. Bergshoeff, I.G. Koh and E. Sezgin *Yang-Mills Einstein supergravity in seven dimensions* Phys. Rev. D, **32**, 6, (1985) 1353.
- [25] R. D'Auria and P. Fre, *Geometric Supergravity in d = 11 and Its Hidden Supergroup*, Nucl. Phys. B **201** (1982) 101 [Nucl. Phys. B **206** (1982) 496].

- [26] P. Fré, *Comments on the 6-index photon in $d = 11$ supergravity and the gauging of free differential algebras* Class. Quantum Gravity, **1** L81 (1984).
- [27] Pietro G. Fré *Gravity: a Geometrical Course*, Volume One and Volume Two, Springer 2013
- [28] G. James and M. Liebeck, *Representations and Characters of Groups*, Cambridge Mathematical TextBooks, Cambridge University Press (1993)
M. Hamermesh, *Group Theory and its applications to Physical Problems* Addison and Wesley Pub. Co. 1964.
- [29] Saad Adnan, *A characterization of $PSL(2, 7)$* Journ. London Math. Soc. (2), 13 (1976), 215-225.
- [30] L. Castellani, R. D'Auria, P. Fré, *Supergravity and String Theory: a geometric perspective*, World Scientific, book in three volumes (1990).
- [31] G. Dall'Agata, D. Fabbri, C. Fraser, P. Fre, P. Termonia and M. Trigiante, *The $Osp(8-4)$ singleton action from the supermembrane*, Nucl. Phys. B **542** (1999) 157 doi:10.1016/S0550-3213(98)00765-2 [hep-th/9807115].